Problems 1–3 due Monday, 2 Nov. 2015; Problems 4–6 due Monday, 9 Nov. 2015.

Problems

1: Please find an explicit union of annuli containing the roots of

$-60600+5051206x+100949994x^2-7060505x^3+100405x^4+1009701x^5-20101x^6$.

(The radii should be rational linear combinations of logs of integers.) The theorem on locating roots of exponential sums in an explicit union of vertical strips (stated in class on 22 Oct.) would be very useful here.

2: Suppose $g\in\mathbb{C}[x_1,\ldots,x_N]$ is not identically zero and define $Z_{\mathbb{C}}(g)\subseteq\mathbb{C}^N$ to be the set of complex roots of g. Please show that $\mathbb{C}^N\setminus Z_{\mathbb{C}}(g)$ is path-connected.

Hint: Try reducing to the special case $N=1$, which you've already solved in HW#2.

Note: The statement fails utterly if we assume $g\in\mathbb{R}[x_1,\ldots,x_n]$ and instead work with $Z_{\mathbb{R}}(g)$: Consider $\mathbb{R}\setminus Z_{\mathbb{R}}(x_1)$. A conceptual reason things are different over $\mathbb{C}$ is dimension-doubling: $\mathbb{C}^N$ has *real* dimension $2N$ and $Z_{\mathbb{C}}(g)$ has *real* dimension at most $2N-2$. So, roughly speaking, this problem is asking you to prove that path-connectedness can *not* be destroyed by removing an algebraic set of (real) codimension ≥ 2.

3: Suppose $c_0,\ldots,c_n\in\mathbb{R}^*$, $b_1,\ldots,b_n\in\mathbb{R}$, and $a_2,\ldots,a_n\in\mathbb{R}^n$ are linearly independent vectors. Please find, and prove, a tight upper bound on the maximal number of isolated roots in $\mathbb{R}^n_+$ of the system of equations

$$\begin{aligned} c_0+c_1x_1^{b_1}+\cdots+c_nx_n^{b_n}&=0\\ x^{a_2}-1&=0\\ &\vdots\\ x^{a_n}-1&=0 \end{aligned}$$

Hint: If you make a clever monomial change of variables $x=y^M$, then you can make $x_2,\ldots,x_n$ be functions of a single parameter. You can then substitute into the first equation and reduce to an earlier theorem.

4: Please prove the following version of **Schwartz's Lemma**: If $f\in\mathbb{C}[x_1,\ldots,x_n]\setminus\{0\}$ is a polynomial of degree d and $S\subset\mathbb{C}$ is any subset of finite cardinality N, then f vanishes at at most dN^{n-1} points of S^n.

Hint: The $n=1$ case follows immediately from the Weak Fundamental Theorem of Algebra. For general n, you can proceed by induction, and the key trick is to consider f as a polynomial in x_n with coefficients in $\mathbb{C}[x_1,\ldots,x_{n-1}]$.

5: (a) Please prove that the Archimedean tropical variety of $1+x+y$ is contained in the amoeba of $1+x+y$.

Hint: You can actually parametrize the boundary of the amoeba explicitly by observing that $|y|=|1+x|$ in the underlying zero set and then applying the triangle inequality.

(b) Please compute the Archimedean tropical variety of $(x+1)^3+y$.

6: *Khovanski's Theorem on Real Fewnomials*, derived in the early 1980s, was the first definitive generalization of Descartes' Rule to higher dimensions. A *very* special case implies that the 2×2 system of equations

$$(\star) \quad F := (f_1, f_2) := \begin{Bmatrix} c_{1,1} + c_{1,2}x^{a_2} + c_{1,3}x^{a_3} \\ c_{2,1} + c_{2,2}x^{a_4} + c_{2,3}x^{a_5} \end{Bmatrix}$$

for $x = (x_1, x_2)$, $c_{i,j} \in \mathbb{R}$ and $a_i \in \mathbb{R}^2$, has at most 5184 non-degenerate roots in $\mathbb{R}^2_+$. You'll carry out a short, elementary proof of a tighter upper bound of 6. (The best one can do is 5, thanks to a 2003 paper of myself and Li and Wang.)

(a) First show how to find a rescaling of variables $(x_1, x_2) = (\gamma_1 y_1, \gamma_2 y_2)$ (and a rescaling of the equations) to reduce to the special case where $f_1 = 1 \pm y_1 \pm y_2$.
Hint: If you know how to solve binomial systems then this is very easy.

(b) Show that, for any homogeneous polynomial $p \in \mathbb{R}[S_1, S_2]$ of degree D, we have

$$\frac{d}{dt}\left[p(t, 1-t)t^\alpha(1-t)^\beta\right] = q(t, 1-t)t^{\alpha-1}(1-t)^{\beta-1}$$

for some homogeneous $q \in \mathbb{R}[S_1, S_2]$ of degree $\leq D + 1$ and $A, B \in \mathbb{R}$.

(c) Show that any function of the form $g(t) := 1 + At^a(1-t)^b + Bt^c(1-t)^d$ has at most 6 roots in the open interval $(0, 1)$.
Hint: Use Rolle's Theorem to show that, in the open interval $(0, 1)$, g has at most 1 more root than g'. By applying Part (b), and dividing out by a suitable monomial term, you'll then get an expression of the form $g_{1,1} + g_{1,2}$ where $g_{1,1}$ is a degree 1 polynomial in t and $g_{1,2}$ is a monomial in $(t, 1-t)$ times a degree 1 polynomial in t. Repeating this a few times, you'll get more complicated expressions that ultimately reduce to 0.

(d) Put (a), (b), and (c) together to prove F has ≤ 6 non-degenerate roots in $\mathbb{R}^2_+$.

NOTE: Please feel free to e-mail comments, questions, and/or corrections.