Problems 1–3 due Monday, 2 Nov. 2015; Problems 4–6 due Monday, 9 Nov. 2015.

Problems

1: Please find an explicit union of annuli containing the roots of

$$f(x) := -60600 + 5051206x + 100949994x^2 - 7060505x^3 + 100405x^4 + 1009701x^5 - 20101x^6$$

(The radii should be rational linear combinations of logs of integers.) The theorem on locating roots of exponential sums in an explicit union of vertical strips (stated in class on 22 Oct.) would be very useful here.

Solution:

Use a software, for example Maple, to do the computation. We determine the polygon $ArchNewt(f)$:

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with(simplex):
ArchNewtf := {[0, -evalf(ln(60600))], [1, -evalf(ln(5051206))], [2, -evalf(ln(100949994))],
[3, -evalf(ln(7060505))], [4, -evalf(ln(100405))], [5, -evalf(ln(1009701))], [6, -evalf(ln(20101))]};
convexhull(P);
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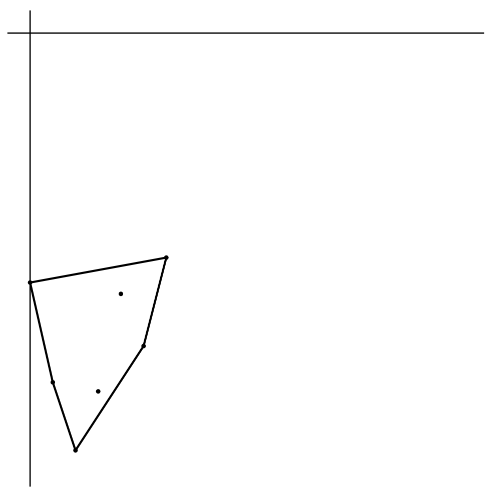
Approximately,

$$ArchNewtf := convhull\{[0, -11.012], [1, -15.435], [2, -18.430], [3, -15.770],$$

$$[4, -11.517], [5, -13.825], [6, -9.909]\}$$

We get $covexhull(ArchNewtf) = convexhull\{v_0, v_1, v_2, v_5, v_6\}$,



As $Archtrop(f) = \{w \in \mathbb{R} | (w, -1)$ is outer-normal to an edge of$ArchNewt(f)\}$, we get

$$Archtrop(f) = \{-ln(5051206) + ln(60600), -ln(100949994) + ln(5051206),$$
$$\frac{1}{3}(-ln(1009701) + ln(100949994)), -ln(20101) + ln(1009701)\}$$

and approximately:

$$Archtrop(f) = \{-4.42, -2.99, 4.61, 3.92\}$$

From Ergür, Paouris and Rojas's Theorem follows that the solutions of $f(e^z)=0$ are contained in the union of strips

$$\bigcup_{w\in Archtrop(f)} (w-log(6), w+log(6))\times\mathbb{R}$$

being $6=\frac{t-1}{\delta(f)}=\frac{7-1}{1}$, and $\delta(f)=min_{i\neq j}|a_i-a_j|$. Hence the solutions to $f(x)=0$ are all contained in the union of annuli

$$\bigcup_{w\in Archtrop(f)} (e^{w-log(6)}, e^{w+log(6)})$$

where $e^{w-log(6)}$ and $e^{w+log(6)}$ are the inner and the outer radii of the annulus.
In this case the solutions to $f(e^z)=0$ are contained in two strips

$$\sim \Big((-6.21,-1.2)\cup(-0.26,5.71)\Big)\times\mathbb{R}$$

and the solutions to $f(x)=0$ are contained in the union of annuli of approximated radii (respectively, inner and outer) $(0.002, 0.3)$ and $(0.77, 301.87)$.

2: Suppose $g\in\mathbb{C}[x_1,\dots,x_N]$ is not identically zero and define $Z_{\mathbb{C}}(g)\subseteq\mathbb{C}^N$ to be the set of complex roots of g. Please show that $\mathbb{C}^N\setminus Z_{\mathbb{C}}(g)$ is path-connected.

Solution:

Let $y,z\in\mathbb{C}^N\setminus Z_{\mathbb{C}}(g)$. Then the set $S:=\{\lambda y+(1-\lambda)z \mid \lambda\in\mathbb{C}\}$ defines a complex line connecting y,z. As the polynomial $g(\lambda y+(1-\lambda)z)=:\bar{g}(\lambda)$ is a nonzero polynomial in one variable ($\bar{g}$ is not identically 0 as for example it does not vanish on $\lambda=1$, being $y\notin Z_{\mathbb{C}}(g)$), then it has finitely many roots in $\mathbb{C}$, which are the points of $S\cap Z_{\mathbb{C}}(g)$. Thus we can reduce this problem to exercise 4, homework 2, and find a path connecting y,z in $S\setminus S\cap Z_{\mathbb{C}}(g)\cong\mathbb{R}^2\setminus X$, where X is a finite subset of $\mathbb{R}^2$.

3: Suppose $c_0,\dots,c_n\in\mathbb{R}^*$, $b_1,\dots,b_n\in\mathbb{R}$, and $a_2,\dots,a_n\in\mathbb{R}^n$ are linearly independent vectors. Please find, and prove, a tight upper bound on the maximal number of isolated roots in $\mathbb{R}^n_+$ of the system of equations

$$\begin{aligned} c_0+c_1x_1^{b_1}+\dots+c_nx_n^{b_n} &=0\\ x^{a_2}-1=0&\\ \vdots&\\ x^{a_n}-1=0& \end{aligned}$$

Solution:

Let $A:=(a_2^T,\dots,a_n^T)\in\mathbb{R}^{n\times n-1}$ the matrix whose columns are the vectors $a_2,\dots,a_n$. As A has rank $n-1$, then we can transform it with elementary column operation (i.e. multiplying it by right with a product of elementary matrices E) so that, for some $\alpha_1,\dots,\alpha_{n-1}\in\mathbb{R}$

$$AE=\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_{n-1}\\ 1 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ \vdots & \dots & \ddots & \vdots\\ 0 & 0 & \dots & 1\end{pmatrix}$$

Hence the last $n-1$ equations of the system $x^A = (1, \ldots, 1)$ can be rewritten as $x^{AE} = (1, \ldots, 1)^E = (1, \ldots, 1)$, and this implies $x_2 = x_1^{-\alpha_1}, \ldots, x_n = x_1^{-\alpha_{n-1}}$.
Substituting into the first equation, we get

$$c_0 + c_1 x_1^{b_1} + c_2 (x_1^{-\alpha_1})^{b_2} + \cdots + c_n (x_1^{-\alpha_{n-1}})^{b_n} = 0$$

which is a polynomial in x_1 only, with $t \leq n+1$ terms: from exercise 2, of homework 2 it has at most $t-1$ positive roots. Substituting back into $x_i = x_1^{-\alpha_{i-1}}$, we get that the system has at most n solutions in $\mathbb{R}_+^n$.

NOTE: Please feel free to e-mail comments, questions, and/or corrections.