4: Please prove the following version of **Schwartz's Lemma**: If K is any field and $f \in K[x_1, \ldots, x_n]$ is a polynomial of degree $d \geq 1$ and $S \subset K$ is any subset of finite cardinality N, then f vanishes at at most dN^{n-1} points of S^n.

Solution:

We proceed by induction on n. For $n = 1$, it follows from the Weak Fundamental Theorem of Algebra that $f \in K[x_1]$ has at most d roots in K, and in particular, it has at most d roots in S.

Assume the statement true for $n - 1$. Consider $f \in K[x_1, \ldots, x_n] = K[x_1, \ldots, x_{n-1}][x_n]$, i.e. we write f as a polynomial in one variable x_n with coefficients in $\mathbb{C}[x_1, \ldots, x_{n-1}]$:

$$f(x) = c_0(x_1, \ldots, x_{n-1}) + c_1(x_1, \ldots, x_{n-1})x_n + \cdots + c_d(x_1, \ldots, x_{n-1})x_n^d$$

We may assume $d < N$ (for otherwise, the result would be trivially true). Let i be the greatest integer such that c_i is *not* identically zero. (Since f is non-constant, we must have $i \geq 0$.) Any $\zeta = (\zeta_1, \ldots, \zeta_n) \in S^n$ at which f vanishes must then fall into one of two exclusive cases:

(I) $c_i(\zeta_1, \ldots, \zeta_{n-1}) = 0$.
(II) $c_i(\zeta_1, \ldots, \zeta_{n-1}) \neq 0$.

Clearly, $\deg c_i \leq d - i$. So by our Induction Hypothesis, there can be no more than $(d - i)N^{n-2}$ points of S^{n-1} at which c_i can vanish. Clearly then, there are no more than $(d - i)N^{n-2} \cdot N = (d - i)N^{n-1}$ points (in S^n) of Type (I).

Now if $c_i(\zeta_1, \ldots, \zeta_{n-1}) \neq 0$, then $f(\zeta_1, \ldots, \zeta_{n-1}, x_n)$ has degree i as a polynomial in x_n and thus vanishes at no more than i choices of x_n. In other words, there are no more than $N^{n-1}i$ points in S^n of Type (II).

Since $(d - i)N^{n-1} + iN^{n-1} = dN^{n-1}$ we are done. ■

5: (a) Please prove that the Archimedean tropical variety of $1 + x + y$ is contained in the amoeba of $1 + x + y$.

(b) Please compute the Archimedean tropical variety of $(x + 1)^3 + y$.

Solution:

(a) Using the notation from the lecture, we have $c_0 = c_1 = c_2 = 1$ and $a_0 = (0, 0)$, $a_1 = (1, 0)$ and $a_2 = (0, 1)$. From the definition of ArchTrop(f), we know

$$\text{ArchTrop}(f) = \{w \in \mathbb{R}^2 \mid \max\{1, e^{w_1}, e^{w_2}\} \text{ is attained at least twice}\}$$

hence

$$\text{ArchTrop}(f) = \{w \in \mathbb{R}^2 \,|w_1 = w_2, w_1 > 0\} \cup \{w \in \mathbb{R}^2 \,|w_1 = 0, w_2 < 0\} \cup \{w \in \mathbb{R}^2 \,|w_2 = 0, w_1 < 0\}$$

On the other hand, consider

$$\text{Amoeba}(f) = \{(\log|x|, \log|y|) \mid x, y \in \mathbb{C}^* \text{ and } f(x, y) = 0\}$$

To understand this amoeba, first observe that $1 + x + y = 0$ implies that $y = -1 - x$ and thus $|y| = |1 + x|$. So by the Triangle Inequality,

$$|x| - 1 \leq |y| \leq |x| + 1.$$

Similarly, $1 = -x - y$ implies that $1 = |x + y|$ and thus

$$1 \leq |x| + |y|$$

again by the Triangle Inequality. So the amoeba is the image of a rectangular strip under the pairwise log-map. Since log is continuous on $\mathbb{R}^2_+$, we see that boundary of the amoeba is the image of the boundary of our strip: The boundary is parametrized by

$$\{(\log s, \log(1-s))\}_{s \in (0,1)} \cup \{(\log s, \log(s-1))\}_{s>1} \cup \{(\log s, \log(s+1))\}_{s>0}.$$

The amoeba thus clearly contains the rays nonnegatively generated by $(1, 0)$, $(0, 1)$, and $(1, 1)$. So our amoeba indeed contains $\text{ArchTrop}(1 + x + y)$.

(b) We have

$$\text{ArchTrop}(f) = \{w \in \mathbb{R}^2 \mid \max\{1, 3e^{w_1}, 3e^{2w_1}, e^{3w_1}, e^{w_2}\} \text{ is attained at least twice}\}$$

The terms can be rewritten as $1, e^{w_1+log(3)}, e^{2w_1+log(3)}, e^{3w_1}, e^{w_2}$. If $w_1 > 0$, then 1 cannot be the maximum.
If $0 < w_1 < log(3)$ then $e^{3w_1} < e^{2w_1+log(3)}$ and $e^{w_1+log(3)} < e^{2w_1+log(3)}$, hence the maximum is attained twice if $w_2 = 2w_1 + log(3)$.
If $w_1 = log(3)$, then the maximum is attained twice at $e^{3w_1}, e^{2w_1+log(3)}$ if $w_2 < 0$.
If $w_1 > log(3)$, then $e^{3w_1} > e^{2w_1+log(3)} > e^{w_1+log(3)}$, hence the maximum can be attained twice just if $w_2 = 3w_1$.
If $w_1 = 0$, then the maximum is attained twice at $e^{2w_1+log(3)}$ and $e^{w_1+log(3)}$, if $w_2 < 0$. Now the case $w_1 < 0$: if $w_1 < -log(3)$, then the maximum is attained twice at e^{w_2} and 1, if $w_2 = 0$. If $w_1 = -log(3)$, then the max is attained twice at $e^{w_1+log(3)}$ and 1, for $w_2 \leq 0$ if $-log(3) < w_1 < 0$ then the maximum is attained at $e^{w_1+log(3)}$ and e^{w_2}, so if $w_2 = w_1 + log(3)$. hence

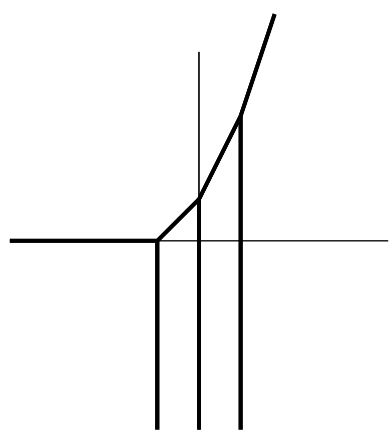

Figure 1: $\text{Archtrop}((1 + x)^3 + y)$

6: *Khovanski's Theorem on Real Fewnomials*, derived in the early 1980s, was the first definitive generalization of Descartes' Rule to higher dimensions. A *very* special case implies that the 2×2 system of equations

$$(\star) \quad F := (f_1, f_2) := \begin{Bmatrix} c_{1,1} + c_{1,2}x^{a_2} + c_{1,3}x^{a_3} \\ c_{2,1} + c_{2,2}x^{a_4} + c_{2,3}x^{a_5} \end{Bmatrix}$$

for $x = (x_1, x_2)$, $c_{i,j} \in \mathbb{R}$ and $a_i \in \mathbb{R}^2$, has at most 5184 non-degenerate roots in $\mathbb{R}^2_+$. You'll carry out a short, elementary proof of a tighter upper bound of 6. (The best one can do is 5, thanks to a 2003 paper of myself and Li and Wang.)

(a) First show how to find a rescaling of variables $(x_1, x_2) = (\gamma_1 y_1, \gamma_2 y_2)$ (and a rescaling of the equations) to reduce to the special case where $f_1 = 1 \pm z_1 \pm z_2$.

(b) Show that, for any homogeneous polynomial $p \in \mathbb{R}[S_1, S_2]$ of degree D, we have

$$\frac{d}{dt}\left[p(t, 1-t)t^{\alpha}(1-t)^{\beta}\right] = q(t, 1-t)t^{\alpha-1}(1-t)^{\beta-1}$$

for some homogeneous $q \in \mathbb{R}[S_1, S_2]$ of degree $\leq D + 1$ and $A, B \in \mathbb{R}$.

(c) Show that any function of the form $g(t) := 1 + At^a(1-t)^b + Bt^c(1-t)^d$ has at most 6 roots in the open interval $(0, 1)$.
Hint: Use Rolle's Theorem to show that, in the open interval $(0, 1)$, g has at most 1 more root than g'. By applying Part (b), and dividing out by a suitable monomial term, you'll then get an expression of the form $g_{1,1} + g_{1,2}$ where $g_{1,1}$ is a degree 1 polynomial in t and $g_{1,2}$ is a monomial in $(t, 1-t)$ times a degree 1 polynomial in t. Repeating this a few times, you'll get more complicated expressions that ultimately reduce to 0.

(d) Put (a), (b), and (c) together to prove F has ≤ 6 non-degenerate roots in $\mathbb{R}^2_+$.

Solution:

(a) First assume that a_2, a_3 are linearly independent column vectors.
Making the monomial change of variables $(x_1, x_2) := (u, v)^{[a_2, a_3]^{-1}}$ makes the first function an affine form in u and v. (This change of variables preserves the number of roots in $\mathbb{R}^2_+$, thanks to an earlier homework problem.) It is then clear that we can put the first function into the form $1 \pm u' \pm v'$ by dividing the first function by $c_{1,1}$ and letting $u = \frac{c_{1,1}}{|c_{1,2}|}u'$ and $v = \frac{c_{1,1}}{|c_{1,3}|}v'$. (This also clearly leaves the number of roots in $\mathbb{R}^2_+$ preserved.)
We can have two cases: either the first equation is of the form $f_1 = 1 - u' - v'$, which would allow us a substitution of the type $v' = 1 - u'$, or $f_1 = 1 + u' - v'$ (the case $f_1 = 1 + u' + v'$ would lead to no positive solutions). In the latter case, we can divide by v' (the number of positive solutions is not affected) and obtain $f_1 = \frac{1}{v'} + \frac{u'}{v'} - 1$. We call $z_1 := \frac{1}{v'}$ and $z_2 := \frac{u'}{v'}$ and obtain an equation of the desired form.

Now, if a_2 and a_3 are linearly dependent, we simply let $u := x^{a_2}$ and $v := x^{a_3'}$ where a_3' is any vector that is not a multiple of a_2. We then obtain that the first function is a trinomial in u and the second function becomes a trinomial in u and v. From hw2, ex2, it follows that the first equation has at most 2 positive solutions, that substituted in the second equation, would give a trinomial in v that has at most 2 solutions, for a total of at most 4 solutions.

(b) Computing

$$\frac{d}{dt}\left[p(t,1-t)t^{\alpha}(1-t)^{\beta}\right] = \frac{d}{dt}[p(t,1-t)]t^{\alpha}(1-t)^{\beta} + p(t,1-t)\frac{d}{dt}(t^{\alpha}(1-t)^{\beta})$$

we can call $F(t,1-t) := \frac{d}{dt}[p(t,1-t)]$, which is a polynomial of degree $\leq D-1$. Then the above expression can be rewritten as

$$F(t,1-t)t^{\alpha}(1-t)^{\beta} + p(t,1-t)t^{\alpha-1}(1-t)^{\beta-1}(\alpha(1-t)-\beta t) =$$

$$= \underbrace{[F(t,1-t)t(1-t) + p(t,1-t)(\alpha-\alpha t-\beta t)]}_{:=q(t,1-t)} t^{\alpha-1}(1-t)^{\beta-1}$$

and q is a polynomial of degree $\leq D+1$.

(c) As g is a differentiable function, we can apply Rolle's Theorem to the intervals defined by adjacent roots of g in $(0,1)$. In particular, g has no more roots in $(0,1)$ than 1 plus the number of roots of g' in $(0,1)$. More generally, we can apply this trick recursively to obtain that g has no more roots in $(0,1)$ than k plus the number of roots of $g^{(k)}$. Note also that since the number of roots of a function in $(0,1)$ is unaffected by multiplying by a function of the form $t^{a}(1-t)^{b}$, we can use $t^{\alpha_1}(1-t)^{\beta_1}\frac{d}{dt}(t^{\alpha_2}(1-t)^{\beta_2}\frac{d}{dt}(\cdots(g)\cdots)$ in place of $g^{(k)}$ in the last statement, where k applications of $\frac{d}{dt}$ are used.

Observe now that

$$\begin{aligned} g' &= Aat^{a-1}(1-t)^{b} - Abt^{a}(1-t)^{b-1} + Bct^{c-1}(1-t)^{d} - Bdt^{c}(1-t)^{d-1} \\ &= At^{a-1}(1-t)^{b-1}(a(1-t)-bt) + Bt^{c-1}(1-t)^{d-1}(c(1-t)-dt). \end{aligned}$$

So then,

$$t^{1-a}(1-t)^{1-b}g' \;=\; A(a-(a+b)t) + Bt^{c-a}(1-t)^{d-a}(c(1-t)-dt)$$

and we can continue taking derivatives, and dividing out by monomials in t and $1-t$. In particular, Part (b) tells us that

$$\frac{d}{dt}(t^{1-a}(1-t)^{1-b}g') \;=\; -A(a+b) + t^{c-a-1}(1-t)^{d-a-1}q(t,1-t)$$

where q is a homogeneous polynomial of degree at most 2. So taking yet another derivative gives us

$$\frac{d^2}{dt^2}(t^{1-a}(1-t)^{1-b}g') \;=\; t^{c-a-2}(1-t)^{d-a-2}r(t,1-t)$$

where r is a homogeneous polynomial of degree at most 3. So then, $t^{2+a-c}(1-t)^{2+a-d}\frac{d^2}{dt^2}(t^{1-a}(1-t)^{1-b}g')$ is a cubic polynomial. So by the Weak Fundamental Theorem of Algebra, and our aforementioned application of Rolle's Theorem, we obtain that g has at most $3+3=6$ roots $(0,1)$.

In order to finish the proof, we just need to put together a,b,c and observe that the positive solutions to $z_1+z_2-1=0$ are in $(0,1)$.