On the Separation of Roots of Univariate Polynomials

J. Maurice Rojas[1]

October 27, 2015

Abstract

This a brief addendum to the proof of the Fundamental Theorem of Algebra presented in class on Tuesday, 27 Oct., 2015. In particular, we explain why the root paths in the proof do not intersect.

Why do the Root Paths Repel?

Following the notation of our proof, recall that $\phi = (\phi_0, \ldots, \phi_d) : [0,1] \longrightarrow \mathbb{C}^{d+1} \setminus \Sigma_d$ is an (analytic) path connecting the degree d polynomials $x^d - 1$ and $f(x)$. In particular, we defined $H(t,x) := \phi_0(t) + \phi_1(t)x + \cdots + \phi_d(t)x^d$, so that $H(0,x) = x^d - 1$ and $H(1,x) = f(x)$. Our proof revolves around showing that the zero set of H in $[0,1] \times \mathbb{C}$ consists of d curves, and that the intersection of these curves with the line $\{(t,x) \mid t = t_0\}$ has cardinality d for any $t_0 \in [0,1]$. A key fact needed in the proof is showing that these curves do not intersect. So we'll show now that the roots of $H(t,x)$ (as t ranges over $[0,1]$) always remain at a distance of at least $\delta > 0$ from each other.

To derive a proof by contradiction, assume that there is a sequence

$$\left(t_1, \zeta^{(1)}, \xi^{(1)}\right), \left(t_2, \zeta^{(2)}, \xi^{(2)}\right), \ldots \in [0,1] \times \mathbb{C} \times \mathbb{C}$$

such that $\lim_{i\to\infty} t_i = \tau$ for some $\tau \in [0,1]$, $\zeta^{(i)} \neq \xi^{(i)}$ and $H\left(t_i, \zeta^{(i)}\right) = H\left(t_i, \xi^{(i)}\right) = 0$ for all i, and $\lim_{i\to\infty} \zeta^{(i)} = \lim_{i\to\infty} \xi^{(i)} = \zeta$ for some $\zeta \in \mathbb{C}$. Note in particular that $H(\tau, \zeta) = 0$ since H is continuous.

By construction, $\frac{\partial H}{\partial x}$ is continuous and well-defined. Clearly then,

$$\lim_{i\to\infty} \frac{H\left(t_i, \zeta^{(i)}\right) - H\left(t_i, \xi^{(i)}\right)}{\zeta^{(i)} - \xi^{(i)}} = \left.\frac{dH}{dx}\right|_{(\tau,\zeta)} = 0.$$

This contradicts the fact that $\phi([0,1])$ avoids Σ_d.

Therefore, the minimum distance between the roots x of $H(t,x)$ (which is a continuous function of t) remains positive for all $t \in [0,1]$. Since $[0,1]$ is compact, this minimum distance function attains a positive minimum δ, and we are done. ■

[1]Department of Mathematics, Texas A&M University TAMU 3368, College Station, Texas 77843-3368, USA. e-mail: `rojas@math.tamu.edu` , web page: `www.math.tamu.edu/~rojas` . Partially supported by NSF grants DMS-1460766 and CCF-1409020, and a von Neumann visiting professorship.