The Univariate Discriminant via the Sylvester Resultant

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Abstract

These notes present a derivation for an explicit lower bound on the minimal spacing between complex roots of a polynomial in one variable. Along the way, we apply Sylvester and van der Monde matrices.

1 Nonzero Resultants Yield Lower Bounds on Separation

We will prove the following result, which implicitly assumes the Fundamental Theorem of Algebra.

Theorem 1.1 *Suppose* $f(x) := c_0 + c_1x + \cdots + c_dx^d \in \mathbb{C}[x]$ *has degree* d *(so* $c_d \neq 0$*) and* d *distinct roots* $\zeta_1, \ldots, \zeta_d$*. Then* $i \neq j$ *implies that*

$$|\zeta_i - \zeta_j| \geq \frac{\sqrt{\left|\mathrm{Res}_{(d,d-1)}(f, f')\right|}}{|c_d|^{d-\frac{1}{2}} 2^{(d+1)(d-2)} \max_{i\in\{0,\ldots,d-1\}} |c_i/c_d|^{\frac{(d+1)(d-2)}{2(d-i)}}} > 0$$

In particular, if we also have that $f \in \mathbb{Z}[x]$, $|c_i| \leq 2^h$ *for all* i *then we can assert a more succinct lower bound of*

$$|\zeta_i - \zeta_j| \geq \left(\frac{1}{2}\right)^{\frac{(d^2+d-3)h}{2}+(d+1)(d-2)} = \left(\frac{1}{2}\right)^{O(d^2h)}.$$

Remark 1.2 *The Fundamental Theorem of Algebra enters in the very first inequality: the fact that having* d *distinct roots implies that* $\mathrm{Res}_{(d,d-1)}(f, f') \neq 0$*. Later we will see a refinement of the above theorem giving a positive lower bound even when* $\mathrm{Res}_{(d,d-1)}(f, f') = 0$. $\diamond$

A key result we'll need is the following algebraic identity.

Lemma 1.3 *Following the notation of Theorem 1.1, we have*

$$\mathrm{Res}_{(d,d-1)}(f, f') = (-1)^{d(d-1)/2} c_d^{2d-1} \prod_{1\leq i<j\leq d} (\zeta_i - \zeta_j)^2.$$

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We will first prove Theorem 1.1 assuming Lemma 1.3, then prove Lemma 1.3 afterward.
Proof of Theorem 1.1: The key is to start with the equality

$$(\star) \qquad \left| c_d^{2d-1} \prod_{1\leq i<j\leq d} (\zeta_i - \zeta_j)^2 \right| = \left|\det \mathrm{Syl}_{(d,d-1)}(f, f')\right|$$

(which is a direct consequence of Lemma 1.3), and find a clever lower bound for the left-hand side. In particular, note that the product of differences on the left-hand side has exactly $d(d-1)/2$ factors, and thus

$$(\star\star) \qquad \prod_{1\leq i<j\leq d} |\zeta_i - \zeta_j| \leq \min_{1\leq i<j\leq d} |\zeta_i - \zeta_j| \left(\max_{1\leq i<j\leq d} |\zeta_i - \zeta_j| \right)^{\frac{d(d-1)}{2}-1}$$

This is clear because the minimal value of $|\zeta_i - \zeta_j|$ must occur at least once, and all the other factors are bounded above by the maximal value of $|\zeta_i - \zeta_j|$. Now, within a disk centered at the origin, the maximal spread between any points is clearly no more than twice the radius. So by Cauchy's Bound,

$$(\star\star\star) \qquad \max_{1\leq i<j\leq d} |\zeta_i - \zeta_j| \leq 2 \cdot 2 \max_{i\in\{0,\ldots,d-1\}} |c_i/c_d|^{1/(d-i)}$$

Combining $(\star)$, $(\star\star)$, and $(\star\star\star)$ we then obtain

$$\left|\det \mathrm{Syl}_{(d,d-1)}(f, f')\right| = \left| c_d^{2d-1} \prod_{1\leq i<j\leq d} (\zeta_i - \zeta_j)^2 \right| \leq |c_d|^{2d-1} \min_{1\leq i<j\leq d} |\zeta_i - \zeta_j|^2 \left(4 \max_{i\in\{0,\ldots,d-1\}} |c_i/c_d|^{1/(d-i)} \right)^{2\left(\frac{d(d-1)}{2}-1\right)}$$

So comparing the left and right quantities above and isolating the "min" term, we obtain

$$\min_{1\leq i<j\leq d} |\zeta_i - \zeta_j|^2 \quad \geq \quad \frac{\left|\det \mathrm{Syl}_{(d,d-1)}(f, f')\right|}{|c_d|^{2d-1} \left(4 \max_{i\in\{0,\ldots,d-1\}} |c_i/c_d|^{1/(d-i)}\right)^{(d+1)(d-2)}}$$

Taking square roots and simplifying slightly, we obtain the first asserted bound.

To obtain the second bound (which assumes the c_i to be integers), we merely observe that $2^h \geq |c_d| \geq 1$ and $|\det \mathrm{Syl}_{(d,d-1)}(f, f')| \geq 1$. So we clearly have

$$\begin{aligned}
\min_{1\leq i<j\leq d} |\zeta_i - \zeta_j| \quad &\geq \quad \frac{1}{|c_d|^{d-\frac{1}{2}} 2^{(d+1)(d-2)} \max_{i\in\{0,\ldots,d-1\}} |c_i|^{\frac{(d+1)(d-2)}{2(d-i)}}} \\
&\geq \quad \frac{1}{|c_d|^{d-\frac{1}{2}} 2^{(d+1)(d-2)} (2^h)^{\frac{(d+1)(d-2)}{2(d-(d-1))}}} \\
&\geq \quad \frac{1}{(2^h)^{d-\frac{1}{2}} 2^{(d+1)(d-2)} (2^h)^{\frac{(d+1)(d-2)}{2}}}.
\end{aligned}$$

Simplifying the denominator, we obtain our second stated bound. ■

2 Discriminants as Products of Differences of Roots

Here we at last prove Lemma 1.3. Toward this end, we first work out some preliminary results. The key ingredients we need are identities involving sums of powers of roots of polynomials and the determinants of some related structured matrices.

Definition 2.1 *For any complex numbers $z_1,\ldots,z_d$, let $V(z_1,\ldots,z_d)$ denote the $d\times d$ matrix*

$$\begin{bmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_d \\ \vdots & \ddots & \vdots \\ z_1^{d-1} & \cdots & z_d^{d-1} \end{bmatrix},$$

usually called a van der Monde matrix. $\diamond$

Lemma 2.2 *For any complex $z_1,\ldots,z_d$, we have* $\det V(z_1,\ldots,z_d)=\prod\limits_{d\geq i>j\geq 1}(z_i-z_j)$.

Proof: We proceed by induction on d. The cases $d\in\{1,2\}$ are trivial, so let us assume the lemma for any fixed value of $d\geq 2$. To conclude, we must then prove that $\det V(z_1,\ldots,z_{d+1})=\prod\limits_{d+1\geq i>j\geq 1}(z_i-z_j)$. This equality is clearly true when $\{z_1,\ldots,z_{d+1}\}$ has cardinality $<d+1$ (since both sides of the equality then vanish), so let us assume $\{z_1,\ldots,z_{d+1}\}$ has cardinality $d+1$.

Defining $p(t):=\det V(z_1,\ldots,z_d,t)$, it is clear that the polynomial p has degree $\leq d+1$, since we can expand the determinant by minors along the last column. Moreover, p clearly vanishes when $t\in\{z_1,\ldots,z_d\}$, for then $V(z_1,\ldots,z_d,t)$ has a repeating column. Clearly then, $p(t)=c\prod\limits_{d\geq j\geq 1}(t-z_j)$, where c is the coefficient of t^{d+1}. This coefficient is clearly $V(z_1,\ldots,z_d)$, so by the induction hypothesis, we obtain $p(t)=\left(\prod\limits_{d\geq i>j\geq 1}(z_i-z_j)\right)\left(\prod\limits_{d\geq j\geq 1}(t-z_j)\right)$. Substituting $t=z_{d+1}$, we are done. ■

Definition 2.3 *Suppose $f\in\mathbb{C}[x_1]$. Then, for any nonnegative integer i, we define the* $i^{\underline{\text{th}}}$ Newton sum *as $N_i:=\sum\limits_{f(\zeta)=0}\mu(\zeta)\zeta^i$, where $\mu(\zeta)$ is the multiplicity of the root ζ.* $\diamond$

Lemma 2.4 *Following the notation of Definition 2.3, write $f(x)=c_0+c_1x+\cdots+c_dx^d$. Then, for any i, we have*

$$(d-i)c_{d-i}=c_dN_i+\cdots+c_{d-i}N_0.$$

Proof: Writing $f(x)=c_d\prod\limits_{f(\zeta)=0}(x-\zeta)^{\mu(\zeta)}$, we clearly have $\frac{f'(x)}{f(x)}=\sum\limits_{f(\zeta)=0}\frac{\mu(\zeta)}{x-\zeta}$. Now

$$\frac{1}{x-\zeta}=\frac{1/x}{1-\frac{\zeta}{x}}=\frac{1}{x}\sum_{i=0}^{\infty}\left(\frac{\zeta}{x}\right)^i=\sum_{i=0}^{\infty}\frac{\zeta^i}{x^{i+1}}.$$

So $\frac{f'(x)}{f(x)} = \sum_{f(\zeta)=0} \mu(\zeta) \sum_{i=0}^{\infty} \frac{\zeta^i}{x^{i+1}} = \sum_{f(\zeta)=0} \frac{1}{x^{i+1}} \sum_{i=0}^{\infty} \mu(\zeta)\zeta^i = \sum_{f(\zeta)=0} \frac{N_i}{x^{i+1}}$, and thus $f'(x) = \left(\sum_{i=0}^{\infty} \frac{N_i}{x^{i+1}}\right) f(x)$. The coefficient of x^{d-i-1} on either side of the last equality then yields $(d-i)c_{d-i}$ (on the left) or $N_0 c_{d-i} + \cdots + N_i c_d$ (on the right). So we are done. ■

Definition 2.5 *Following the notation of Lemma 2.4, let* $\mathcal{P}(f)$ *be the* $d \times d$ *matrix*

$$\begin{bmatrix} N_0 & N_1 & N_2 & \cdots & & N_{d-1} \\ N_1 & N_2 & & & \iddots & N_d \\ N_2 & & & & \iddots & \vdots \\ \vdots & & & & & \\ & \iddots & \iddots & & & \\ N_{d-1} & N_d & \cdots & & & N_{2d-2} \end{bmatrix}.$$ ◇

Lemma 2.6 *Following the notation of Lemma 2.4, let* $\zeta_1, \ldots, \zeta_d$ *denote the roots of* f*, counting multiplicities. Then* $\det \mathcal{P}(f) = \prod_{1 \leq i < j \leq d} (\zeta_i - \zeta_j)^2$.

Proof: Simply note that $\mathcal{P}(f) = V(\zeta_1, \ldots, \zeta_d) V^T(\zeta_1, \ldots, \zeta_d)$, where $(\cdot)^T$ denotes matrix transpose. So then $\det \mathcal{P}(f) = (\det V(\zeta_1, \ldots, \zeta_d))^2$ and we are done by Lemma 2.2. ■

We are at last ready to proof Lemma 1.3.

Proof of Lemma 1.3: Let D denote the $(2d-1)\times(2d-1)$ matrix $\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & N_0 & \cdots & N_{d-1} \\ & & & N_0 & N_1 & \cdots & N_d \\ & & \iddots & & & & \vdots \\ & \iddots & & & & & \vdots \\ N_0 & \cdots & & & & \cdots & N_{2d-2} \end{bmatrix}$, where the first $d-1$ rows are the first $d-1$ standard basis vectors of $\mathbb{R}^{2d-1}$. Also let D' denote the upper-triangular $(2d-1)\times(2d-1)$ matrix $\begin{bmatrix} c_d & \cdots & c_0 & & & & \\ & c_d & \cdots & c_0 & & & \\ & & \ddots & & \ddots & & \\ & & & c_d & \cdots & c_1 & c_0 \\ & & & & c_d & \cdots & c_1 \\ & & & & & \ddots & \vdots \\ & & & & & & c_0 \end{bmatrix}$, which has exactly d super-diagonals. It is then easily checked (via matrix multiplication in blocks) that DD' is exactly the $(2d-1)\times(2d-1)$ matrix $\begin{bmatrix} c_d & \cdots & c_0 & & & & \\ & c_d & \cdots & c_0 & & & \\ & & \ddots & & \ddots & & \\ & & & c_d & \cdots & c_1 & c_0 \\ & & & & dc_d & \cdots & c_1 \\ & & & dc_d & \cdots & c_1 & 0 \\ & & \iddots & & \iddots & & \\ dc_d & \cdots & c_1 & & & & \end{bmatrix}$ (thanks to Lemma 2.4), where the top $d-1$ rows (resp. bottom d rows) correspond to the coefficients of f (resp. f'). Also, it is clear that $\det D = \det \mathcal{P}(f)$ and $\det D' = c_d^{2d-1}$.

To conclude, observe that if we reverse the order of the top $d-1$ rows, we obtain a left-right reflection of the matrix $\mathrm{Syl}_{(d,d-1)}$. Note also that reversing the entries of any k-vector can be done by exactly $1+2+\cdots+(k-1)=\frac{k(k-1)}{2}$ transpositions. Thus, via exactly $(d-1)(d-2)/2$ row swaps and exactly $(2d-1)(2d-2)/2=(d-1)(2d-1)$ column swaps applied to DD', we can obtain the matrix $\mathrm{Syl}_{(d,d-1)}$. Noting that

$$\frac{(d-1)(d-2)}{2}+(d-1)(2d-1)=\frac{(d-1)(5d-4)}{2}$$

has the same parity as $\frac{(d-1)(5d-4)}{2}-\frac{4(d-1)^2}{2}=\frac{d(d-1)}{2}$, we then obtain

$$\det(DD')=(-1)^{d(d-1)/2}\mathrm{Res}_{(d,d-1)}(f,f')=(\det D)(\det D').$$

So by Lemma 2.2, we are done. ■

3 Acknowledgements

The proof of Lemma 1.3 is inspired by the more general framework of sub-discriminants and sub-resultants, as described in [BPR06] (especially around Page 112). [RS02] also develops resultants and discriminants, but less explicitly. However, [RS02] has a better description of Cauchy's Bound and other more much more intricate bounds on the roots of polynomials.

References

[BPR06] Basu, Saugata; Pollack, Ricky; and Roy, Marie-Francoise, *Algorithms in Real Algebraic Geometry,* Algorithms and Computation in Mathematics, vol. 10, Springer-Verlag, 2006.

[RS02] Rahman, Qazi Ibadur; and Schmeisser, Gerhard, *Analytic Theory of Polynomials,* Clarendon Press, London Mathematical Society Monographs 26, 2002.