Scheduling on an unreliable machine

Machine changes its speed in an online fashion. (unexpected speed change or even breakdown)

Model: machine capacity function $\mu: \mathbb{R}_+ \to \mathbb{R}_+$ non-decreasing, continuous function

$\mu(t)$ is the total processing volume that can be completed up to time t

($\mu'(t)$ is the speed at t ; if it exists)

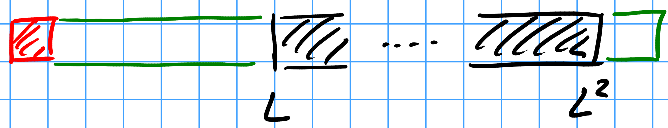
$\boxed{1 \| \sum w_j C_j}$ "ideal machine" : $\mu(t) = t$, $\forall t$

↳ Smith's Rule optimal $\frac{w_j}{p_j} \searrow$

Question: How bad is Smith's Rule if machine changes speed? BAD!

Example: $w_1 = 1+\varepsilon$, $p_1 = 1$

$w_2 = L$, $p_2 = L$



Smith $\quad 1(1+\varepsilon) + (L^2+1)L = \Theta(L^3)$

OPT $\quad L^2 + (L^2+1)(1+\varepsilon) = \Theta(L^2)$

ratio ↗ with $L \to \infty$

Lower bound on OPT

Def. Given a job permutation π and machine capacity function μ, let $S(\pi,\mu)$ be the corresponding schedule when running jobs in order of π on the machine with machine capacity function μ.
Total weight of jobs that have not finished by time t in $S(\pi,\mu) =: S$ is

$$W^S(t) := \sum_{j : C_j^S > t} w_j .$$

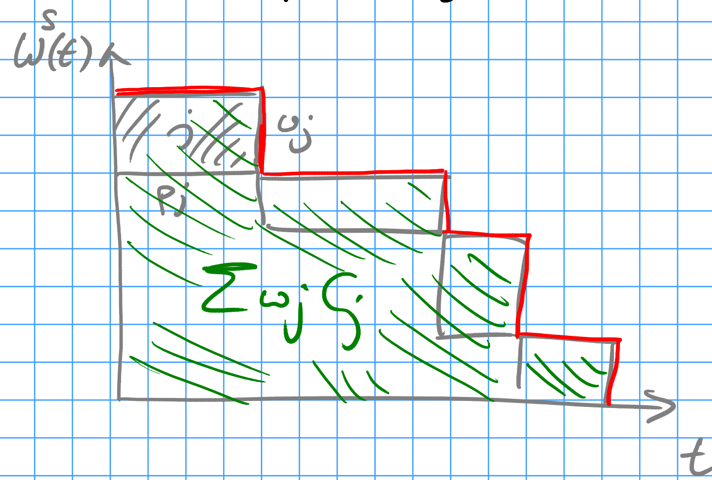
Observe that

$$\sum_j w_j C_j^S = \sum_j w_j \int_0^\infty x_j^S(t)\,dt ,$$

$x_j^S(t)$ indicates if j has completed by t or not

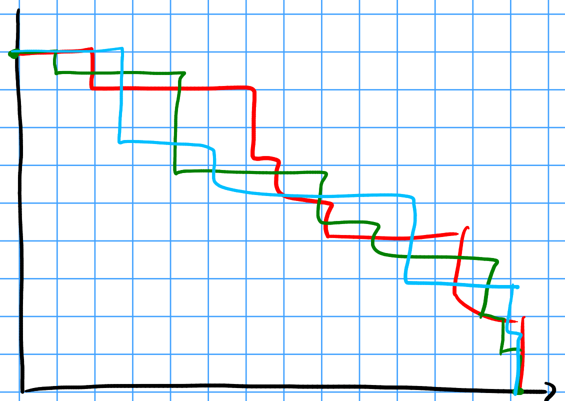
$$= \int_0^\infty \sum_j w_j x_j^S(t)\,dt$$

$$= \int_0^\infty W^S(t)\,dt .$$



__Observation__ Given μ, for $t \geq 0$, let $W^{S^*(\mu)}(t) := \min_{\pi} W^{S(\pi,\mu)}(t)$.

Then
$$\int_0^\infty W^{S^*(\mu)}(t)\, dt$$
is a lower bound on the objective value of any schedule.



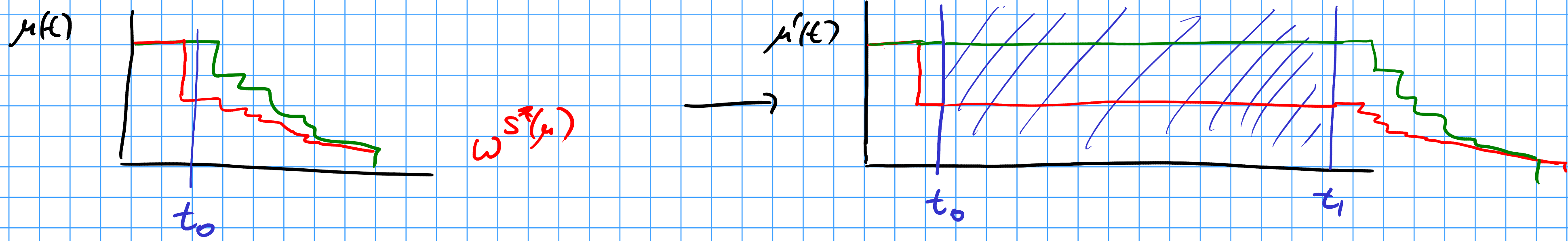
"several π's"

__Key Lemma__: Let π be a sequence of jobs and $c > 0$. Then $\sum w_j C_j^{S(\pi,\mu)} \leq c \cdot OPT(\mu)$ if and only if
$$W^{S(\pi,\mu)}(t) \leq c \cdot W^{S^*(\mu)}(t), \quad \forall t \geq 0 \text{ and each } \mu.$$

__Proof__: "$\Rightarrow$" clear by observation above.

$\longrightarrow$

"$\Leftarrow$": Assume that $W^{s(\pi,\mu)}(t_0) > c \cdot W^{s^*(\mu)}(t_0)$, for some t_0 and μ.



Formally

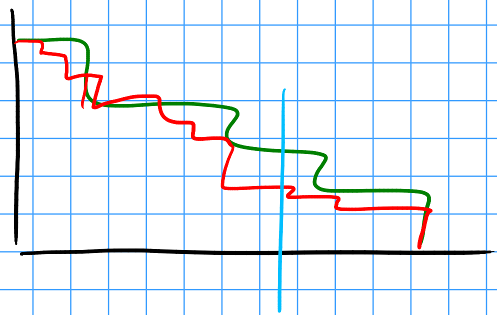
$$\mu'(t) = \begin{cases} \mu(t) & t \le t_0 \\ \mu(t_0) & t_0 < t \le t_1 \\ \mu(t - t_1 + t_0) & t > t_1 \end{cases}$$

Then $$\sum w_j C_j^{s(\pi,\mu')} = \sum w_j C_j^{s(\pi,\mu)} + (t_1 - t_0)\, W^{s(\pi,\mu')}(t_0) = a$$

but there exists a sequence π^* with $W^{s(\pi^*,\mu)}(t_0) = W^{s^*(\mu')}(t_0)$

$$\sum w_j C_j^{s(\pi^*,\mu')} = \sum w_j C_j^{s(\pi^*,\mu)} + (t_1 - t_0) \cdot W^{s^*(\mu')}(t_0) = b$$

As $t_1 \to \infty$, then $a/b > c$ which is a contradiction. $\square$



Observation: sufficient to think of ideal machine $\mu(t)=t,\ \forall t$

$$W^{S(\pi,\mu)}(t) = W^{\pi}(\mu(t))$$

ideal machine

Assume $w_j \in [0,1]$. $w_j \geq 1$.

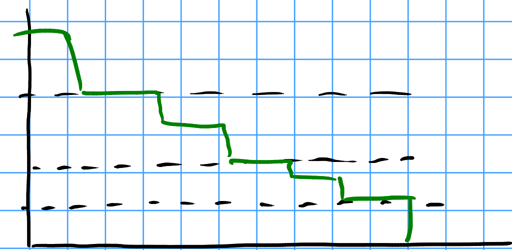
DOUBLING Algorithm

1) For $i = 0, \dots, \lceil \log w(J) \rceil$:

Find subset $J_i^* \subseteq J$ of maximum $p(J_i^*)$, s.t. $w(J_i^*) \leq 2^i$.

2) Construct π from the end; place J_0^* last, J_1^* before that, etc.

unscheduled jobs of

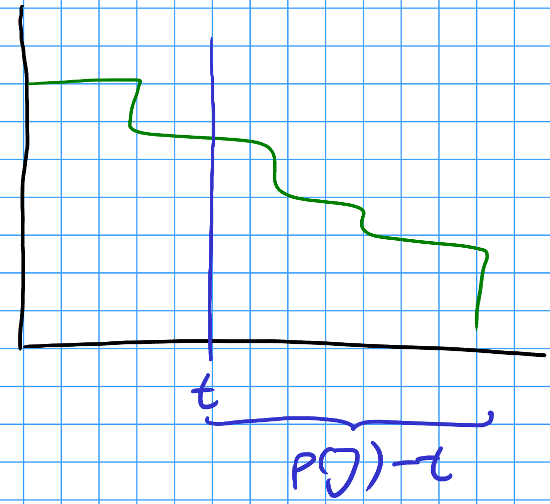


Theorem: DOUBLING produces a universal 4-approximation.

↳ one sequence at most factor 4 from OPT for any μ

Proof. By key lemma it is sufficient to show $W^{\pi}(t) \le 4 \cdot W^{*}(t)$.

Let $t \ge 0$. Let i be minimal s.t. $p(J_i^*) \ge p(J) - t$.



$$\Rightarrow \quad W^{\pi}(t) \le \sum_{k=0}^{i} w(J_k^*) \le \sum_{k=0}^{i} 2^k = 2^{i+1} - 1$$

OPT: By our choice of i: $p(J_{i-1}^*) < p(J) - t$

$$\Rightarrow \quad W^*(t) > 2^{i-1}$$

$$\Rightarrow \quad \frac{W^{\pi}(t)}{W^*(t)} < 4$$

□

Running time ?!
- Need to solve a knapsack problem → pseudopolyn. time
- May turn this into polyn. time alg. adopting a knapsack FPTAS.

→ Instead of J_i^*, we find J_i with $w(J_i) \le (1+\varepsilon)2^i$ and $p(J_i) \ge \max\{p(J') \mid J' \subseteq J \text{ with } w(J') \le 2^i\}$.

$\Rightarrow$ $(4+\varepsilon)$-approximation

Deterministic alg: 4-approx. is best possible.

Randomized alg: approx. factor $e \approx 2.71$ when "randomizing the doubling factor"

→ Instead of weightbound 2^i choose Xe^i where $X = e^Y$ with $Y \sim [0,1]$ uniformly at random

→ e best possible

[Epstein, Levin, ... SIAM J. on Computing, 2012.]