Approximation Algorithms (MA5517)

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Problem Set 1

Applying Results on the Approximability of the Set Cover Problem

The following negative result on the approximability of the set cover problem will be discussed in the upcoming lecture. It might turn out to be useful for solving some of the problems on this sheet and can be used without proving it.

Theorem 1
Consider the UnweightedSetCover *problem. Let n denote the size of an instance's ground set. There exists some constant $c > 0$ such that if there exists a $(c \ln(n))$-approximation algorithm for* UnweightedSetCover, *then $\mathbb{P} = \mathbb{NP}$.*

Problem 1.1 (Directed Steiner Tree Problem)
In the DirectedSteinerTree problem, we are given as input a directed graph $G = (V, A)$, non-negative costs $c_{v,w} \geq 0$ for each arcs $(v, w) \in A$, a root vertex $r \in V$, and a set of terminals $T \subseteq V$. The goal is to find a minimum-cost tree such that it contains a directed r-v-path for each terminal $v \in T$.

Prove that for some constant c there can be no $(c \ln |T|)$-approximation algorithm for DirectedSteinerTree, unless $\mathbb{P} = \mathbb{NP}$.

Problem 1.2 (Uncapacitated Facility Location Problem)
In the UncapacitatedFacilityLocation problem, we have a set of clients D and a set of facilities F. For each client $d \in D$ and facility $f \in F$, there is a cost $c^A_{d,f} \geq 0$ of assigning client d to facility f. Further, there is a cost $c^F_f \geq 0$ associated with each facility $f \in F$. The goal of the problem is to choose a subset of facilities $F' \subseteq F$ so as to minimize the total cost of the facilities in F' and the cost of assigning the each client $d \in D$ to a facility in F'. In other words, we wish to find $F' \subseteq F$ which minimizes $\sum_{f \in F'} c^F_f + \sum_{d \in D} \min_{f \in F'} c^A_{f,d}$.

a) Show that there exists some $c > 0$ such that there is no $(c \ln |D|)$-approximation algorithm for UncapacitatedFacilityLocation, unless $\mathbb{P} = \mathbb{NP}$.

b) Give an $O(\ln |D|)$-approximation algorithm for UncapacitatedFacilityLocation.

Problem 1.3 (Vertex Cover Problem)
Consider the VertexCover problem for the undirected graph $G = (V, E)$ with weights $w_v \geq 0$ for all vertices $v \in V$.

a) Prove that the set of extreme points of the polyhedron given by the inequalities

$$x_u + x_v \geq 1 \qquad \forall \{u, v\} \in E \tag{1}$$
$$x_v \geq 0 \qquad \forall v \in V \tag{2}$$

is a subset of $\{0, 1/2, 1\}^V$.

Please turn over.

b) Give a 3/2-approximation algorithm for VertexCover when the input graph is planar. You may use the fact that polynomial-time LP solvers return extreme points, and that there is a polynomial-time algorithm to 4-color[1] any planar graph.

Problem 1.4 (Metric Asymmetric Travelling Salesman Problem)
In the MetricAsymmetricTravellingSalesman problem, we are given as input a complete directed graph $G = (V, A)$ with cost $c_{v,w} \geq 0$ for each arc $(v, w) \in A$, such that the arc costs obey the triangle inequality, i.e. $c_{u,w} \leq c_{u,v} + c_{v,w}$ holds for all $u, v, w \in V$. The goal is to find a tour of minimum cost, that is, a directed cycle that contains each vertex exactly once, such that the sum of the cost of the arcs in the cycle is minimized.

One approach to finding an approximation algorithm for this problem is to first find a minimum-cost strongly connected[2] Eulerian[3] subgraph of the input graph. Given a strongly connected Eulerian subgraph of the input to the problem, it is possible to use a technique called "shortcutting" to turn this into a tour of no greater cost by using the triangle inequality. One way to find a strongly connected Eulerian subgraph is as follows:
We first find a minimum mean-cost cycle[4] in the graph. Such a cycle can be found in polynomial time. We then choose one vertex of the cycle arbitrarily, remove all other vertices of the cycle from the graph, and repeat. We do this until only one vertex of the graph is left. Consider the subgraph consisting of all the arcs from all the cycles found.

a) Prove that the subgraph found by the algorithm is a strongly connected Eulerian subgraph of the input graph.

b) Prove that the cost of this subgraph is at most $2H_{|V|}\text{OPT}$, where H_n is the n-th harmonic number and OPT is the cost of an optimal tour. Conclude that this algorithm represents a $2H_{|V|}$-approximation algorithm for MetricAsymmetricTravellingSalesman.

[1] A k-coloring of a graph is an assigment of one of k colors to every vertex such that any pair of adjacent vertices is assigned different colors.

[2] A directed graph is strongly connected if for any pair of vertices $v, w \in V$ there is a directed v-w-path and w-v-path.

[3] A directed graph is Eulerian if the indegree of each vertex equals its outdegree.

[4] A minimum mean-cost cycle is a directed cycle that minimizes the ratio of the cost of the arcs in the cycle to the number of arcs in the cycle.

This problem set will be discussed in the tutorial on November 8th, 2016.