Approximation Algorithms (MA5517)

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Problem Set 5

Semidefinite Programming and the Primal-Dual Method

Problem 5.1 (Maximum 2-SAT)
Semidefinite programming can be used to give improved approximation algorithms for the maximum satisfiability problem. We consider the MAXIMUM2SAT problem, in which every clause has two literals.

a) Express MAXIMUM2SAT as an integer quadratic program in which the only constraints are $y_i \in \{-1, 1\}$ and the objective function is quadratic in the y_i.

   Hint: It may help to introduce a variable which indicates whether the value -1 or 1 means `true`.

b) Derive a 0.878-approximation algorithm for MAXIMUM2SAT.

Problem 5.2 (Minimum Spanning Tree Problem)
Consider the primal-dual algorithm for STEINERFOREST discussed in the lecture.

a) Show that the MINIMUMSPANNINGTREE problem is a special case of STEINERFOREST.

b) Show that for MINIMUMSPANNINGTREE, the primal-dual algorithm is a 1-approximation.

   Hint: It is equivalent to a 1-approximation algorithm for MINIMUMSPANNINGTREE that you already know.

Problem 5.3 (Dijkstra's Algorithm)
Prove that the primal-dual shortest s-t-path algorithm from the lecture is equivalent to Dijkstra's algorithm, that is, in each step, it adds the same edge that Dijkstra's algorithm would add.

Problem 5.4 (Multicut Problem in Trees)
Consider the MULTICUT problem in trees. In this problem, we are given an undirected tree $T = (V, E)$, $k \in \mathbb{N}$ pairs of vertices $s_i, t_i \in V, i \in [k]$, and cost $c_e \geq 0$ for each edge $e \in E$. The goal is to find a minimum-cost set of edges F such that for all $i \in [k]$, s_i and t_i are in different connected components of $G' := (V, E \setminus F)$.

For $i \in [k]$, let P_i be the unique s_i-t_i-path in T. Then we can formulate the problem as the following integer program:

$$\begin{aligned} \min \sum_{e \in E} c_e x_e & & \\ \text{s.t.} \quad \sum_{e \in P_i} x_e &\geq 1 & \text{for all } i \in [k] \\ x_e &\in \{0, 1\} & \text{for all } e \in E \end{aligned}$$

Please turn over.

Suppose we root the tree at an arbitrary vertex $r \in V$. For $v \in V$, let $\text{depth}(v)$ be the number of edges on the r-v-path in T. Let $\text{lca}(s_i, t_i)$ be the lowest common ancestor of s_i and t_i, i.e. the vertex $v \in V$ on P_i whose depth is minimum. Suppose we use the primal-dual method to solve this problem, where the dual variable we increase in each iteration corresponds to the violated constraint that maximizes $\text{depth}(\text{lca}(s_i, t_i))$.

Prove that this gives a 2-approximation algorithm for the MULTICUT problem in trees.

Hint: Clean up.

This problem set will be discussed in the tutorial on January 17th, 2017.