Technische Universität München | Zentrum Mathematik
Lehrstuhl für Operations Research

Approximation Algorithms (MA5517)

Dr. Jannik Matuschke | M.Sc. Marcus Kaiser

Problem Set 6

More on the Primal-Dual Method and Iterated Rounding

Problem 6.1 (Minimum-Cost Branching Problem)
In the MinimumCostBranching problem we are given a directed graph $G = (V, A)$, a root vertex $r \in V$, and weights $w_a \geq 0$ for all arcs $a \in A$. The goal of the problem is to find a minimum-cost set of arcs $B \subseteq A$ such that for every $v \in V$, there is exactly one directed r-v-path in B.

Use the primal-dual method to give an optimal algorithm for this problem.

Hint: Consider strongly connected subgraphs.

Problem 6.2 (k-Median Problem)
Recall the k-Median problem from lecture: We are given as input a set of clients D and a set of facilities F, with metric distances $d_{i,j} \geq 0$ for all facilities $i \in F$ and clients $j \in D$. Further, we are given as input a positive integer k that is an upper bound on the number of facilities that can be opened. The goal is to select a subset of facilities of at most k facilities to open and an assignment of clients to open facilities so as to minimize the total assignment costs.

Show that the optimal solution can be found in polynomial time if the optimum cost is 0.

Problem 6.3 (Survivable Network Design Problem)
Let $G = (V, E)$ be an undirected graph and $c_e \geq 0$ be costs on the edge e for all $e \in E$. For some function $f : 2^V \to \mathbb{N}_0$, consider the linear program

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq f(S) && \text{for all } S \subseteq V \\ & 0 \leq x_e \leq 1 && \text{for all } e \in E. \end{aligned}$$

For the following, assume that f is a supermodular function, i.e. for every $A, B \subseteq V$, it holds that

$$f(A) + f(B) \leq f(A \cap B) + f(A \cup B).$$

Let x be any basic feasible solution to the linear program such that $0 < x_e < 1$ holds for all $e \in E$.

a) Show that if the LP's inequalities given by two subsets $A, B \subseteq V$ are fulfilled with equality by the solution x, then the same is true for the inequalities corresponding to the subsets $A \cup B$ and $A \cap B$, and $\mathbb{1}_{\delta(A)} + \mathbb{1}_{\delta(B)} = \mathbb{1}_{\delta(A \cap B)} + \mathbb{1}_{\delta(A \cup B)}$ holds.

Please turn over.

b) Prove that there exists a collection of subsets $\mathcal{L} \subseteq 2^V$ with the following properties.

i) The collection $\mathcal{L}$ is laminar, i.e. for all $A, B \in \mathcal{L}:\ A \cap B = \emptyset \ \lor\ A \subseteq B \ \lor\ B \subseteq A$.

ii) For every $S \in \mathcal{L}$, the respective inequality of the LP is satisfied by x with equality.

iii) The indicator vectors $\mathbb{1}_{\delta(S)}$ for $S \in \mathcal{L}$ are linearly independent.

iv) $|\mathcal{L}| = |E|$.

Hint: For a collection $\mathcal{L}' \subseteq 2^V$ fulfilling (i) to (iii), and a set $S \subseteq V$, it might be useful to consider the potential

$$\Phi(S) := \left|\{T \in \mathcal{L}' :\ S \cap T \neq \emptyset \ \land\ S \setminus T \neq \emptyset \ \land\ T \setminus S \neq \emptyset\}\right| .$$

This problem set will be discussed in the tutorial on January 31st, 2017.