Warm-up to Lecture 3

(November 14, 2016)

In the next lecture, we will discuss greedy and local search algorithms for spanning trees. In preparation to the lecture, it will be helpful to make yourself familiar with the following fundamental results from graph theory (you probably have heard of them already in other classes).

1 Spanning trees

Let $G = (V, E)$ be a graph. A *spanning tree* of G is a set of edges $T \subseteq E$ such that (V, T) is connected and T does not contain any cycles. The following statements are equivalent:

a) T is a spanning tree of G.

b) $|T| = |V| - 1$ and T does not contain any cycles.

c) $|T| = |V| - 1$ and (V, T) is connected.

d) (V, T) is connected, but $(V, T \setminus \{e\})$ contains exactly two connected components for any $e \in T$.

e) Adding any edge to T creates a unique cycle. (That is, for every $v, w \in V$, there is exactly one cycle in $T \cup \{e_{vw}\}$, where e_{vw} is an edge with endpoints v and w; note that we do not require $e_{vw} \in E$.)

f) For every $v, w \in V$, there is exactly one v-w-path in T.

Try to prove the equivalence. If you have trouble, check out Section 2.2 of the Combinatorial Optimization textbook by Korte and Vygen.

2 Hall's Theorem

Let $G = (V, E)$ be a graph. A *matching* is a set of edges M such that every vertex in V is incident to at most one edge in M. The matching M *covers* a set of vertices $U \subseteq V$, if every vertex of U is incident to an edge in M. For $X \subseteq V$, define the *neighborhood* of X by $\Gamma(X) := \{y \in V : \{x, y\} \in E\}$.
Hall's Theorem states the following:

Theorem (Hall 1935). *Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \dot{\cup} B$. Then G contains a matching covering A if and only if $|\Gamma(X)| \geq |X|$ for all $X \subseteq A$.*

Hall's Theorem will be useful to show an approximation factor for the spanning tree algorithms discussed in the lecture. You can find a proof of the theorem in Section 10.1 of the textbook by Korte and Vygen.