LP rounding

$$P_j := \{i : x_i \in C_j\} \quad N_j := \{i : \neg x_i \in C_j\}$$

$$\Pr[C_j \text{ is sat.}] = 1 - \prod_{i \in P_j}(1-y_i^*) \cdot \prod_{i \in N_j} y_i^* \overset{(I)}{\geq} 1 - \left(\frac{\sum_{i\in P_j}(1-y_i^*) + \sum_{i \in N_j} y_i^*}{|C_j|}\right)^{|C_j|}$$

$$\overset{(II)}{\geq} 1 - \left(1 - \frac{z_j^*}{|C_j|}\right)^{|C_j|} \overset{(III)}{\geq} \Big(1 - \underbrace{\left(1 - \frac{1}{|C_j|}\right)^{|C_j|}}_{\leq \exp(-1)}\Big) z_j^*$$

$$\geq \left(1 - \frac{1}{e}\right) z_j^*$$

$$\Rightarrow \mathbb{E}[ALG] \geq \left(1 - \frac{1}{e}\right) \cdot \sum_{j=1}^{m} w_j z_j^* \geq \left(1 - \frac{1}{e}\right) \cdot OPT \quad \square$$

Geometric vs. Arithmetic Mean:

(I) $\left(\prod_{i=1}^{k} a_i\right)^{\frac{1}{k}} \leq \frac{1}{k}\sum_{i=1}^{k} a_i$ for any $a_1, \dots, a_k \in \mathbb{R}_+$

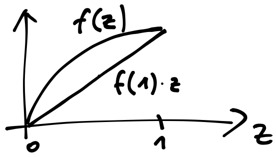
(II) $$\sum_{i \in P_j}(1-y_i^*) + \sum_{i \in N_j} y_i^* = |C_j| - |P_j| - |N_j| - \left(\sum_{i \in P_j}(y_i^* - 1) + \sum_{i \in N_j} -y_i^*\right)$$

$$= |C_j| - \left(\sum_{i\in P_j} y_i^* + \sum_{i \in N_j}(1-y_i^*)\right) \leq |C_j| - z_j^*$$

(III) $f(z) = 1 - \left(1 - \frac{z}{k}\right)^k$ is concave for $z \in [0,1]$.

$f(0) = 0 \quad f(1) = 1 - \left(1 - \frac{1}{k}\right)^k$

$\Rightarrow f(z) \geq \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \cdot z$



Non-linear Randomized Rounding

$$\Pr[C_j \text{ not sat.}] = \prod_{i \in P_j}(1 - f(y_i^*)) \prod_{i \in N_j} f(y_i^*) \leq \prod_{i \in P_j} 4^{-y_i^*} \prod_{i \in N_j} 4^{y_i^* - 1}$$

$$= 4^{-\left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} 1 - y_i^*\right)} \leq 4^{-z_j^*}$$

as in (III)

$$\Pr[C_j \text{ sat}] \geq 1 - 4^{-z_j^*} \geq (1 - 4^{-1}) z_j^* = \frac{3}{4} z_j^*$$

$$\mathbb{E}[ALG] \geq \sum_{j=1}^{m} w_j \Pr[C_j \text{ sat.}] \geq \frac{3}{4}\sum_{j=1}^{m} w_j z_j^* \geq \frac{3}{4} OPT \quad \square$$

Semidefinite Programming for MAX CUT

Lemma Let $x \in \{-1, 1\}^n$. Let $S := \{i \in V : x_i = 1\}$. Then

$$\sum_{e \in \delta(S)} w(e) = \frac{1}{2} \sum_{\{i,j\} \in E} w_{ij} (1 - x_i x_j).$$

Proof: $\{i,j\} \in \delta(S) \iff i \in S, j \notin S$ or $i \notin S, j \in S$

$\iff x_i \neq x_j$

Also: $1 - x_i x_j = \begin{cases} 2 & \text{if } x_i \neq x_j \\ 0 & \text{if } x_i = x_j \end{cases}$ for $x_i, x_j \in \{-1, 1\}$.

Thus: $\sum_{\{i,j\} \in E} w_{ij} (1 - x_i x_j) = 2 \sum_{\{i,j\} \in \delta(S)} w_{ij}$ $\square$

Lemma $OPT \leq Z^*$

Proof: Let $S \subseteq V$. Define $v_i = \begin{pmatrix} x_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ with $x_i := \begin{cases} 1 & \text{if } i \in S \\ -1 & \text{if } i \notin S \end{cases}$.

Then $v_i^T v_i = 1 \quad \forall i \in [n]$ and

$$Z^* \geq \frac{1}{2} \sum_{\{i,j\} \in E} w_{ij} (1 - v_i^T v_j) = \frac{1}{2} \sum_{\{i,j\} \in E} (1 - x_i x_j)$$

$$= \sum_{e \in \delta(S)} w(e). \ \square$$

Proof of approximation guarantee:

Consider $U = \mathrm{span}(v_i^*, v_j^*)$. Let r' be the projection of r on U.

Then $r'' := \frac{r'}{\|r'\|}$ is uniformly distributed on unit circle in U.

$$\mathbb{E}[ALG] = \sum_{\{i,j\} \in E} w_{ij} \cdot Pr[\{i,j\} \in \delta(S)] = \sum_{\{i,j\} \in E} w_{ij} \frac{\arccos(v_i^{*T} v_j^*)}{\pi}$$

$$\geq 0.878 \sum_{\{i,j\} \in E} w_{ij} \frac{1}{2} (1 - v_i^{*T} v_j^*) = 0.878 Z^* \quad \square$$