Lecture: Approximation Algorithms

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Previously ...

Randomized rounding of LPs and SDPs

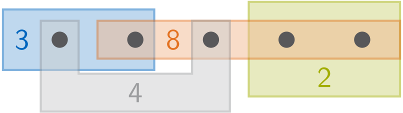
- non-linear rounding (Max Sat)
- Chernoff bounds (Integer Multicommodity Flow)
- selecting a random hyperplane (Max Cut)

The primal-dual method

The SET COVER problem

Input: elements E, sets $\mathcal{S} \subseteq 2^E$, weights $w : \mathcal{S} \to \mathbb{R}_+$

Task: find $\mathcal{S}' \subseteq \mathcal{S}$ with $\bigcup_{S \in \mathcal{S}'} S = E$
minimizing $\sum_{S \in \mathcal{S}'} w(S)$



$$
\begin{aligned}
\min \quad & \sum_{S \in \mathcal{S}} w(S) x(S) & \\
\text{s.t.} \quad & \sum_{S : e \in S} x(S) \geq 1 & \forall\, e \in E \\
& x(S) \geq 0 & \forall S \in \mathcal{S}
\end{aligned}
$$

Primal-dual for Set Cover

$$\begin{aligned} \max \quad & \sum_{e \in E} y(e) & \\ \text{s.t.} \quad & \sum_{e \in S} y(e) \leq w(S) & \forall\, S \in \mathcal{S} \\ & y(e) \geq 0 & \forall e \in E \end{aligned}$$

Algorithm:

1. Initialize $y(e) = 0$ for all $e \in E$.
2. while ($\exists$ uncovered element e')
   Increase $y(e')$ until a set S with $e' \in S$ becomes tight.
   Add S to $\mathcal{S}'$. (increase can be 0)
3. Return $\mathcal{S}'$.

Theorem

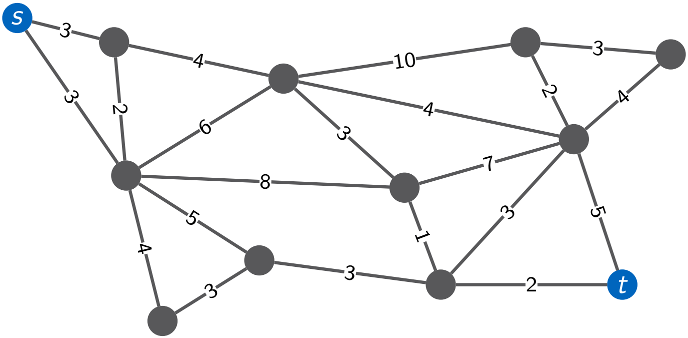
Primal-dual is an f-approximation algorithm for Set Cover.

The shortest path problem

Shortest Path

Input: graph $G = (V, E)$, weights $w : E \to \mathbb{R}_+$,
start $s \in V$, target $t \in V$

Task: find an s-t-path P minimizing $\sum_{e \in P} w_e$



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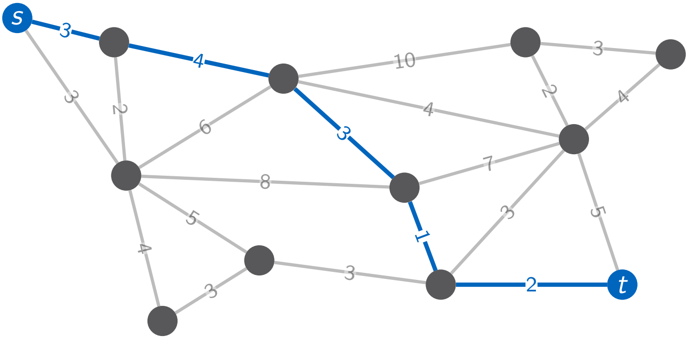
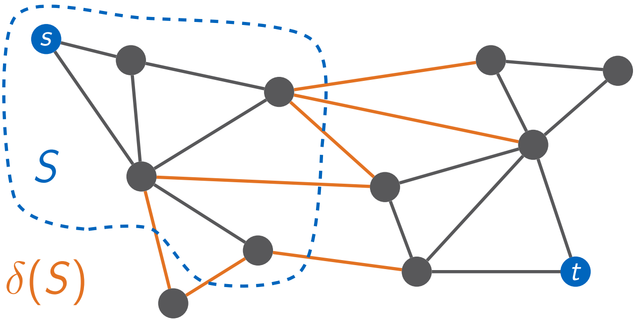


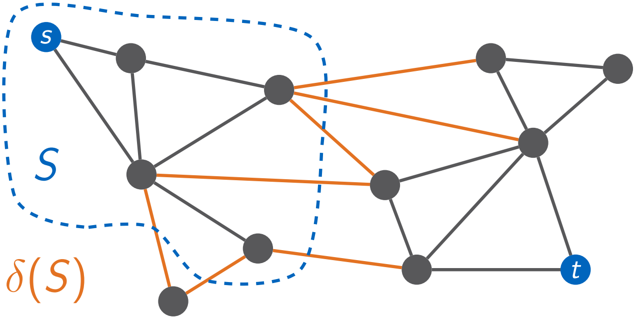
LP relaxation



$$\mathcal{S} := \{S \subseteq V \,:\, s \in S,\, t \notin S\}$$

$$\begin{aligned} \min \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq 1 \qquad \forall\, S \in \mathcal{S} \\ & x_e \in \{0,1\} \qquad \forall\, e \in E \end{aligned}$$

LP relaxation



$$\mathcal{S} := \{S \subseteq V \; : \; s \in S,\; t \notin S\}$$

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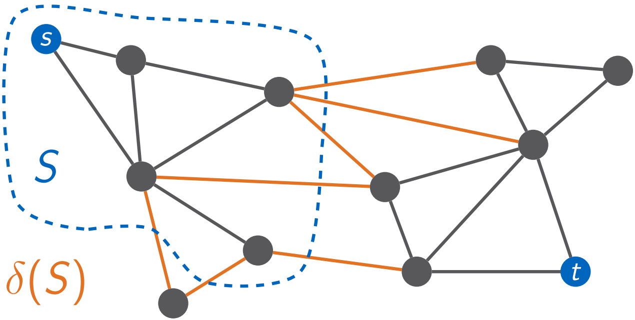

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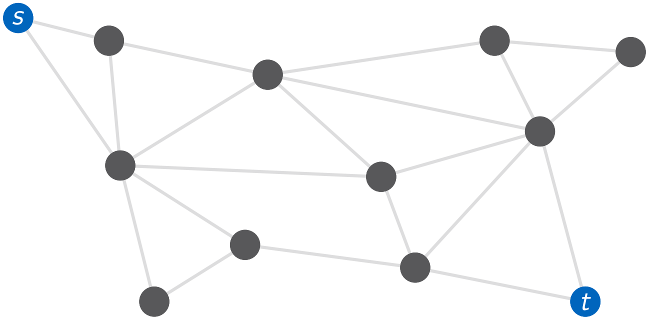
Primal-dual algorithm

Algorithm

1. $T := \emptyset$, $y := 0$
2. while T does not contain s-t-path
   - Let C be the set of vertices connected to s in (V, T).
   - Increase y_C until there is an $e \in \delta(C)$ with $\sum_{e \in \mathcal{S}_e} y_S = w_e$.
   - $T := T \cup \{e\}$ (increase can be 0)
3. Let P be an s-t-path in T.
4. return P

Theorem

Primal-dual is a 1-approximation algorithm for SHORTEST PATH.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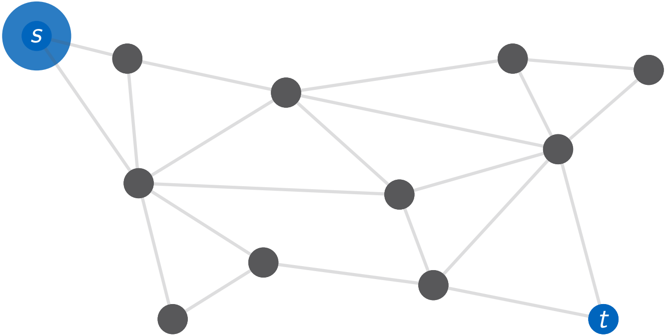
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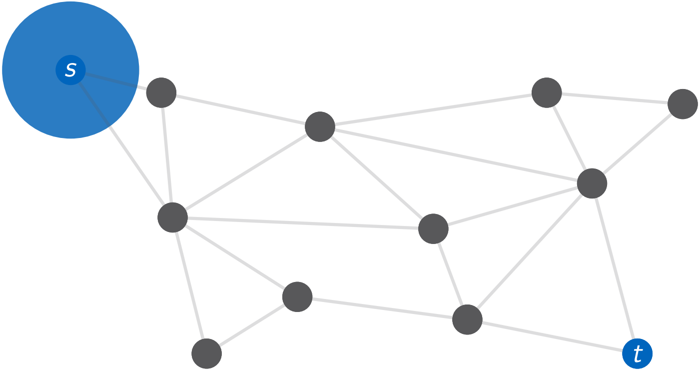
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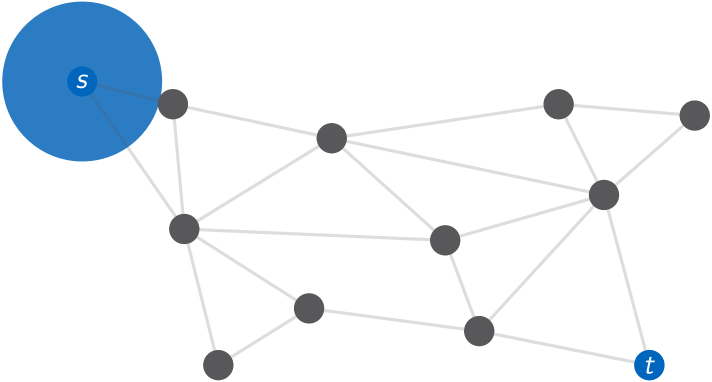
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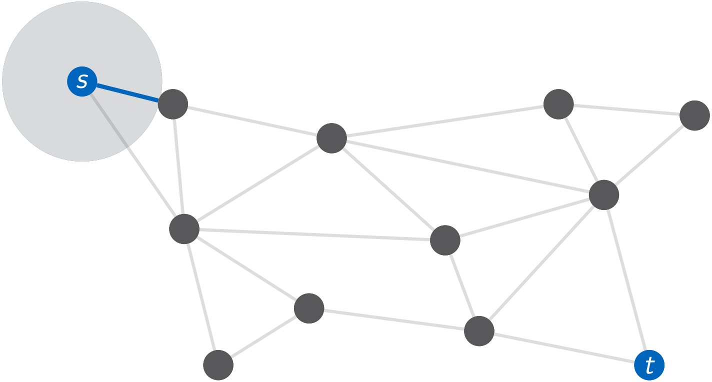
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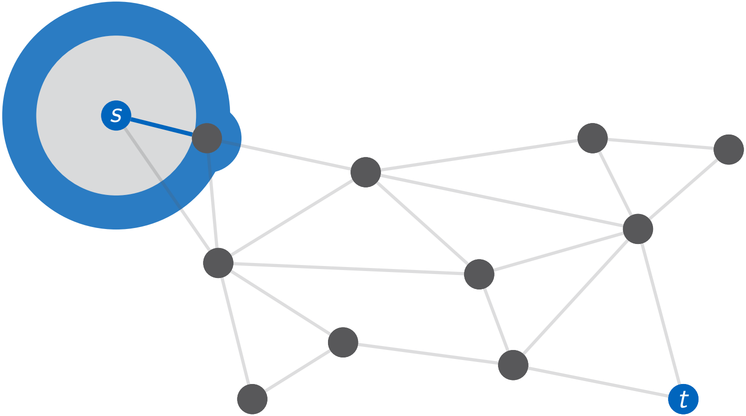
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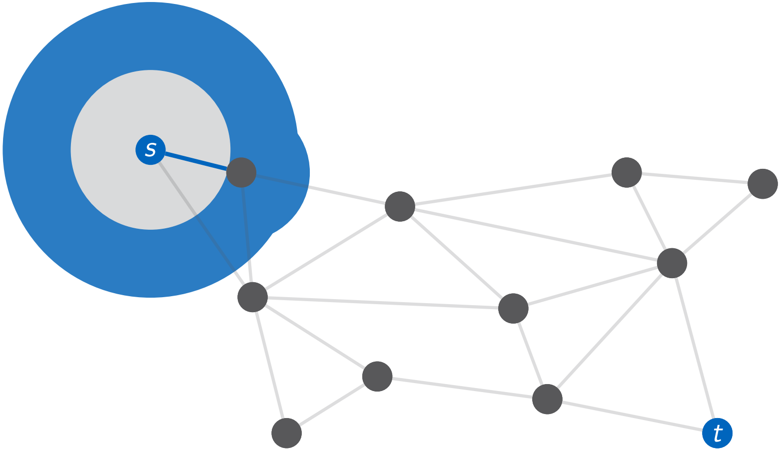
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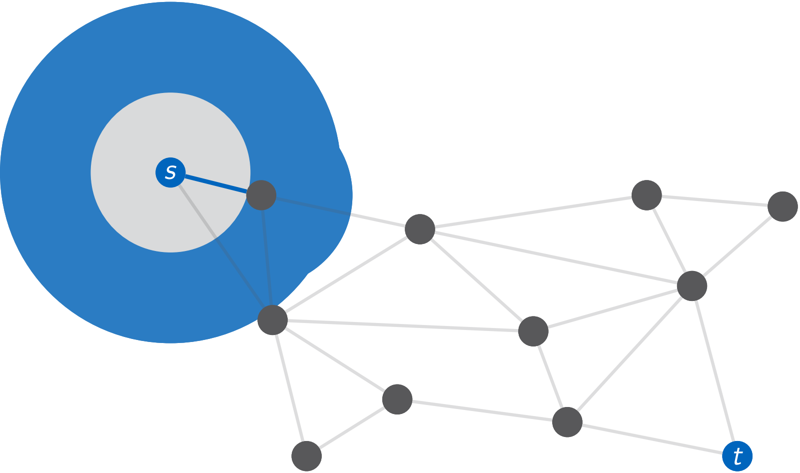
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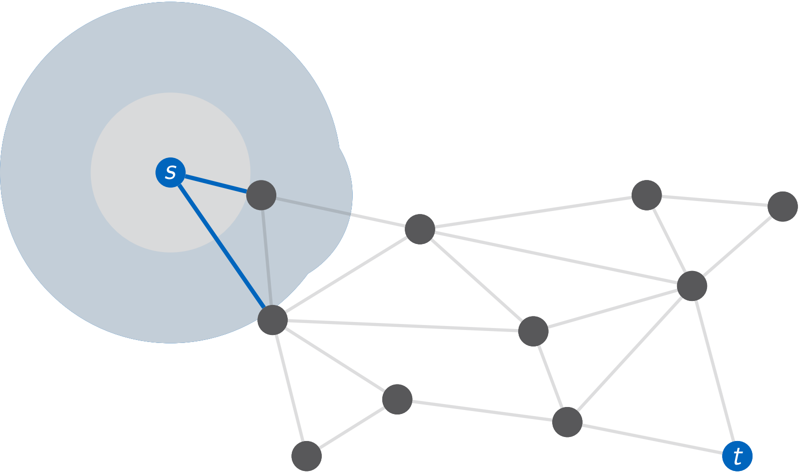

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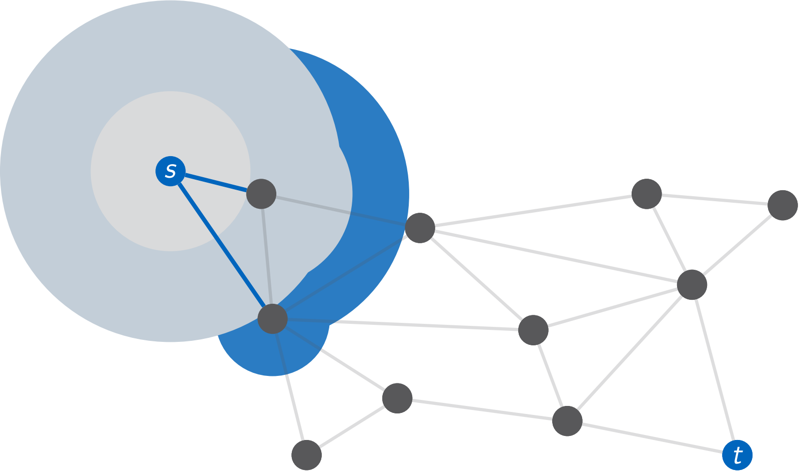
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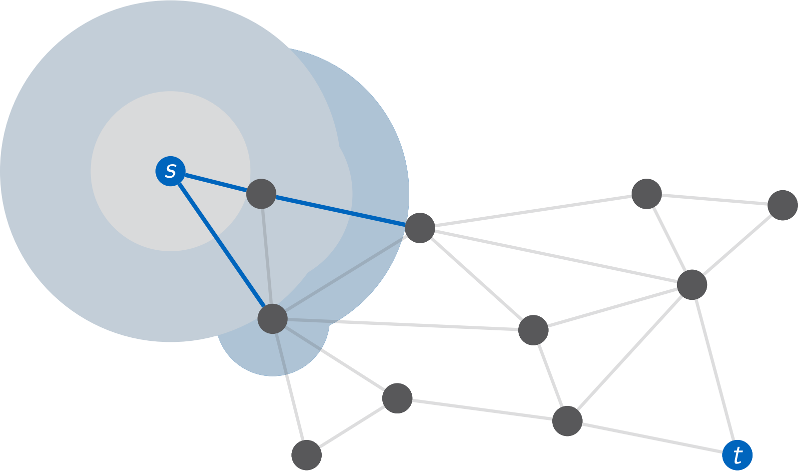
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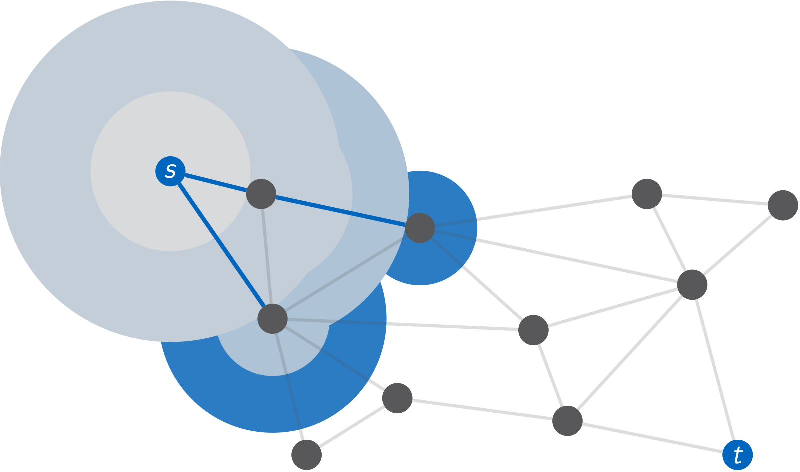
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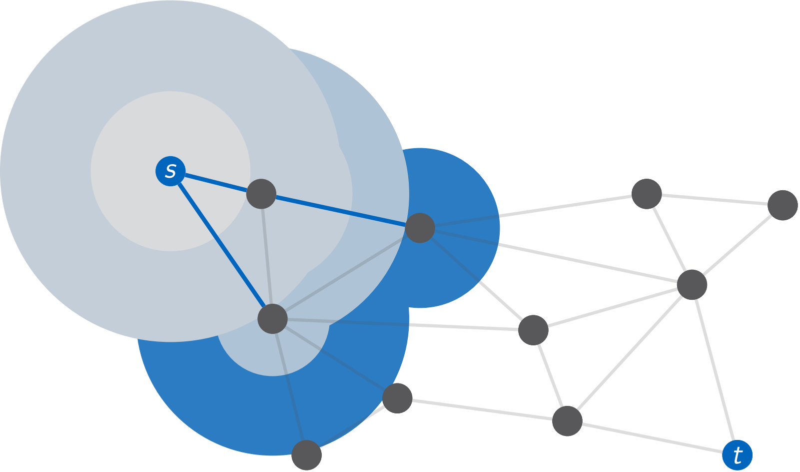
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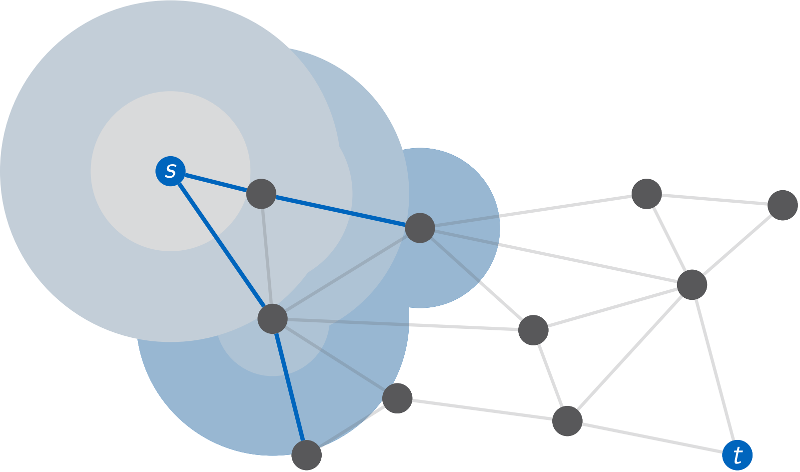
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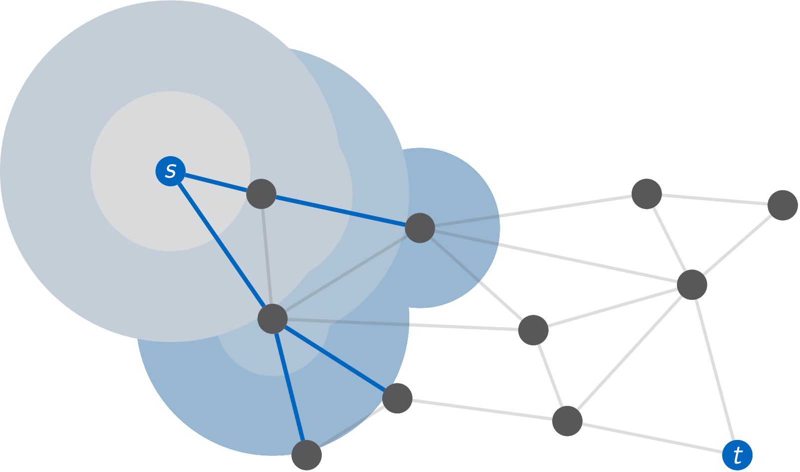
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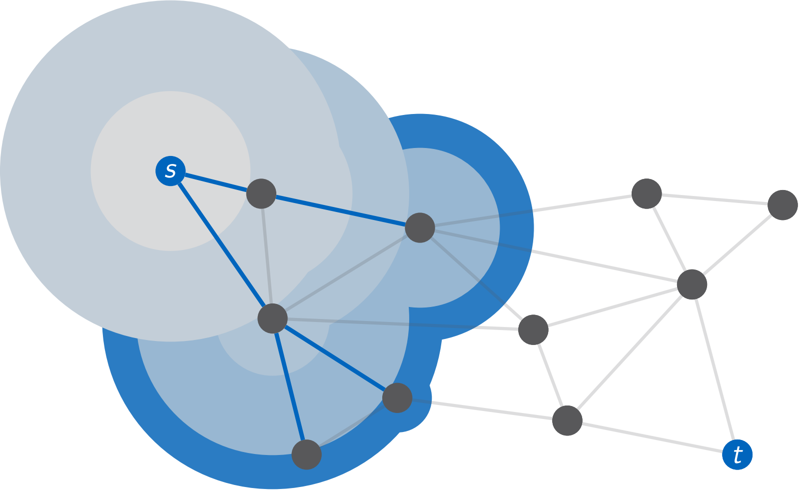
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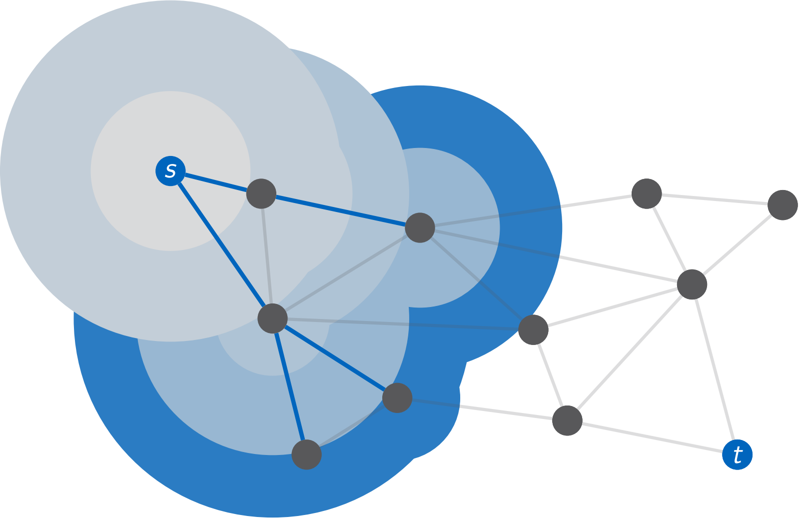
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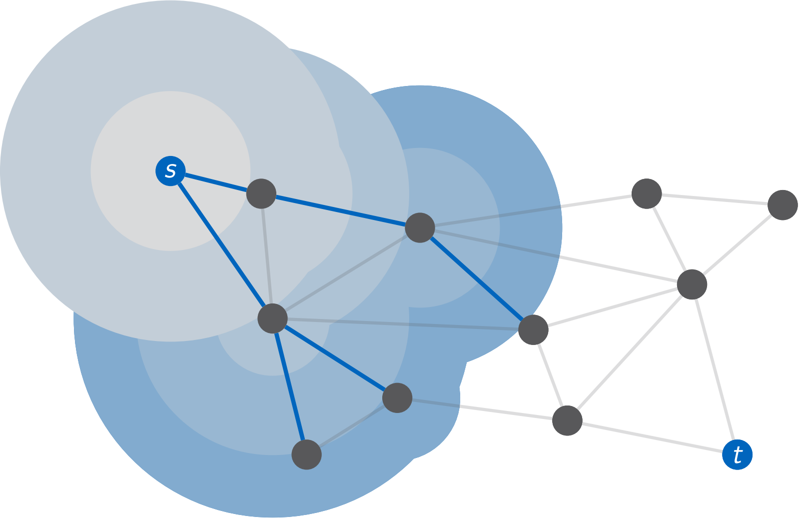
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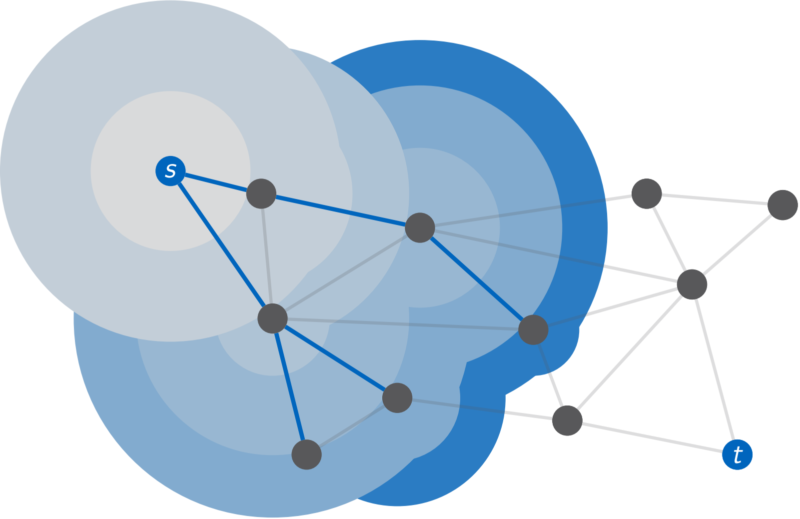
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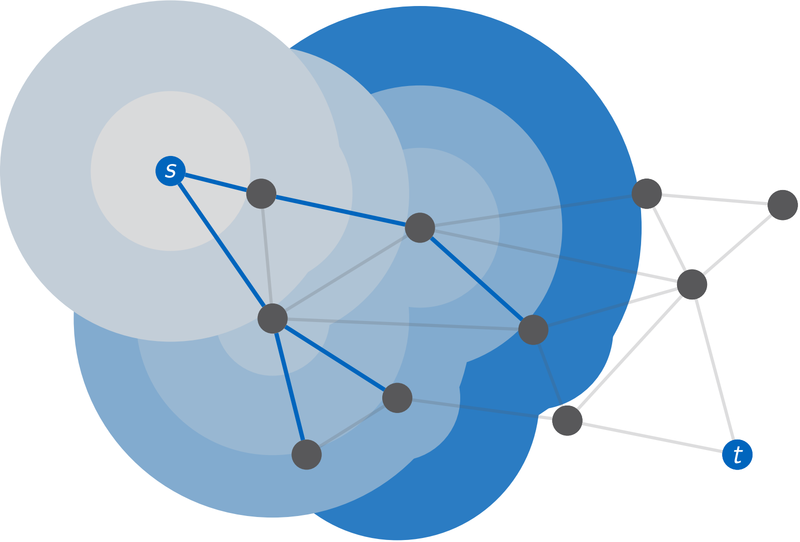
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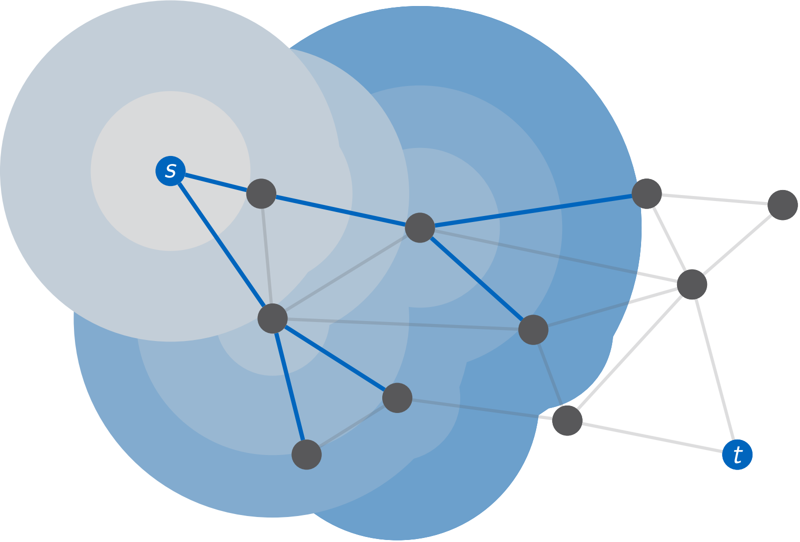
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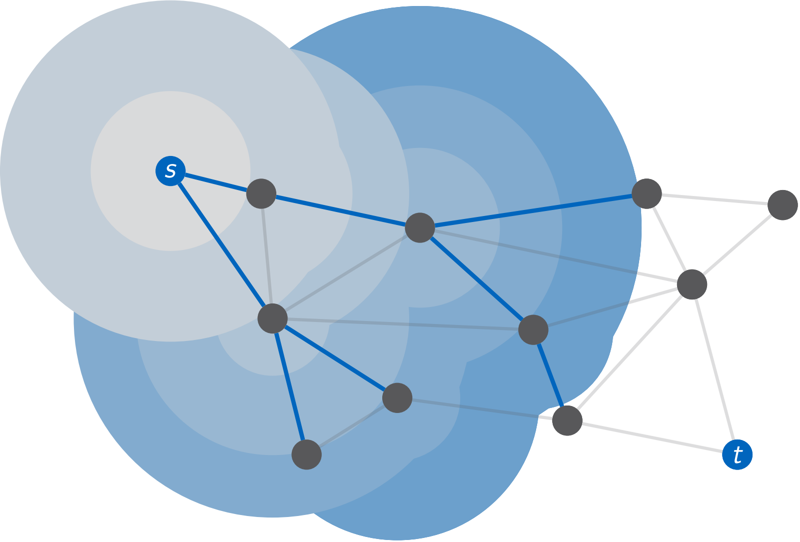

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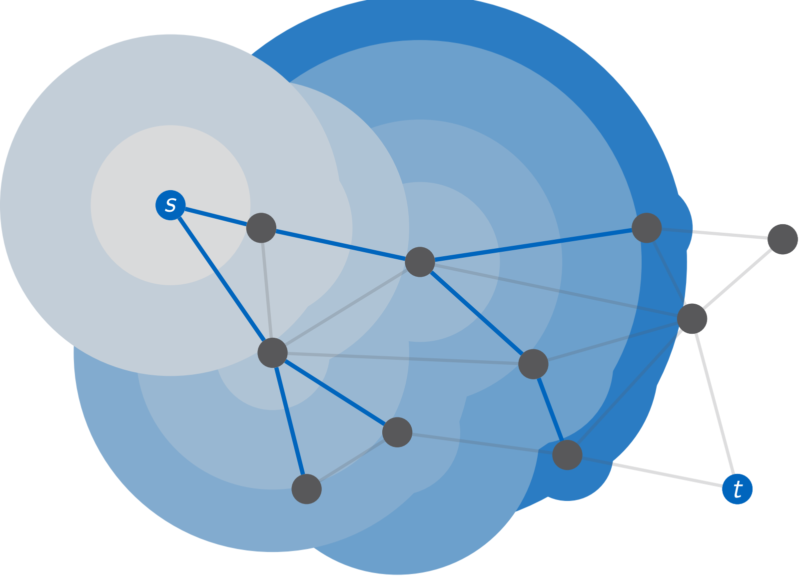

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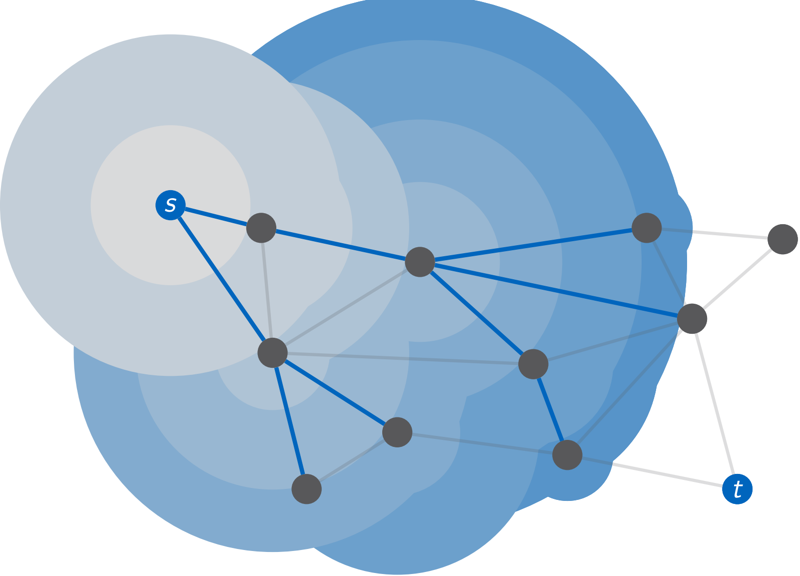
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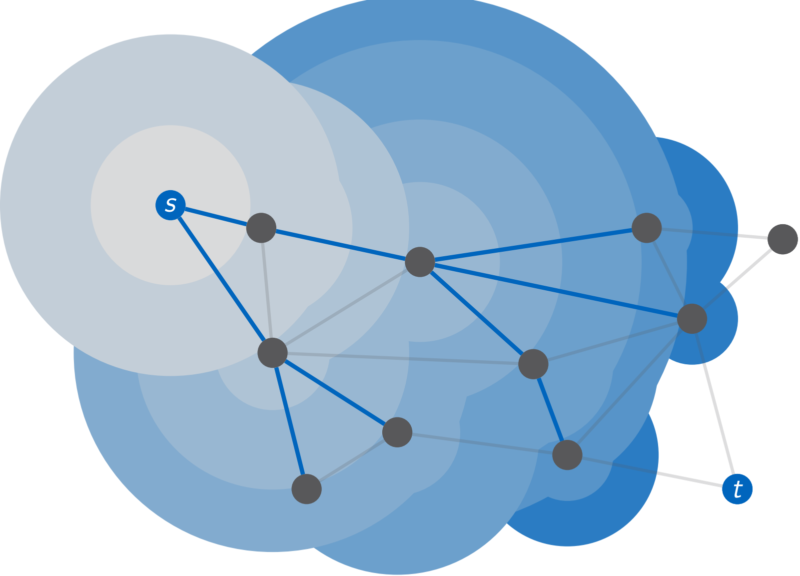
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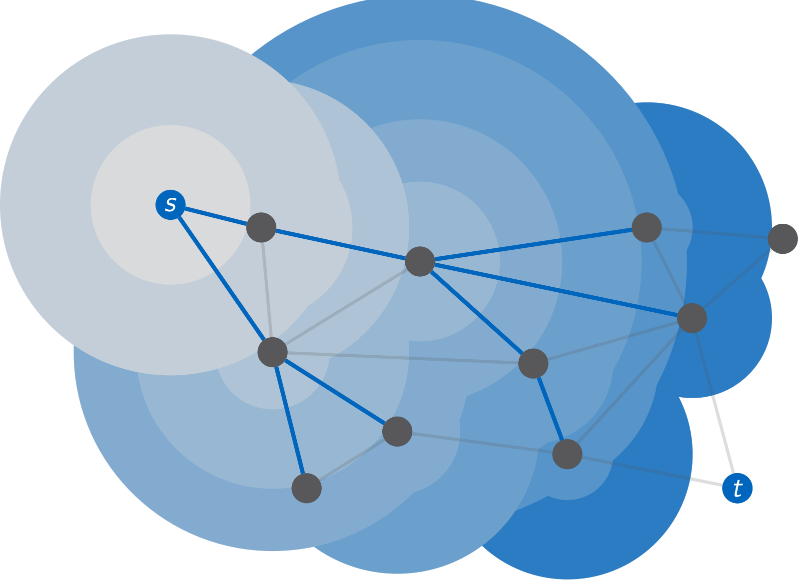
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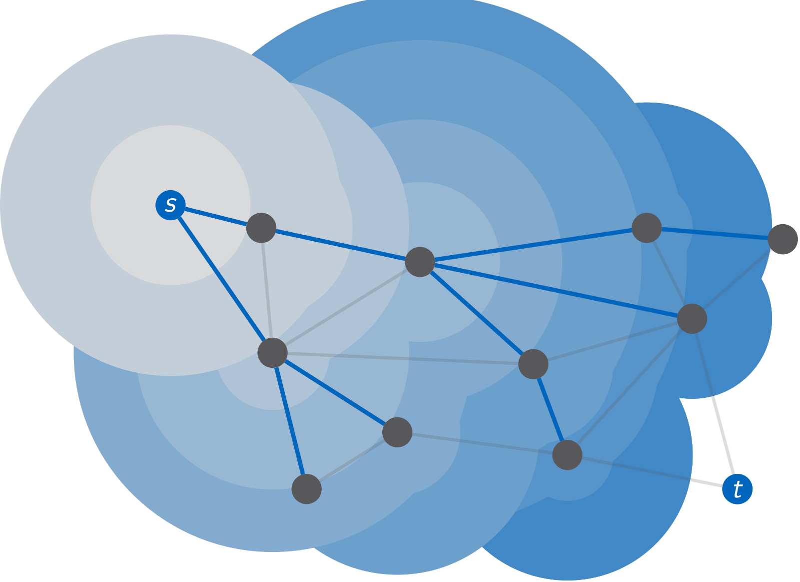
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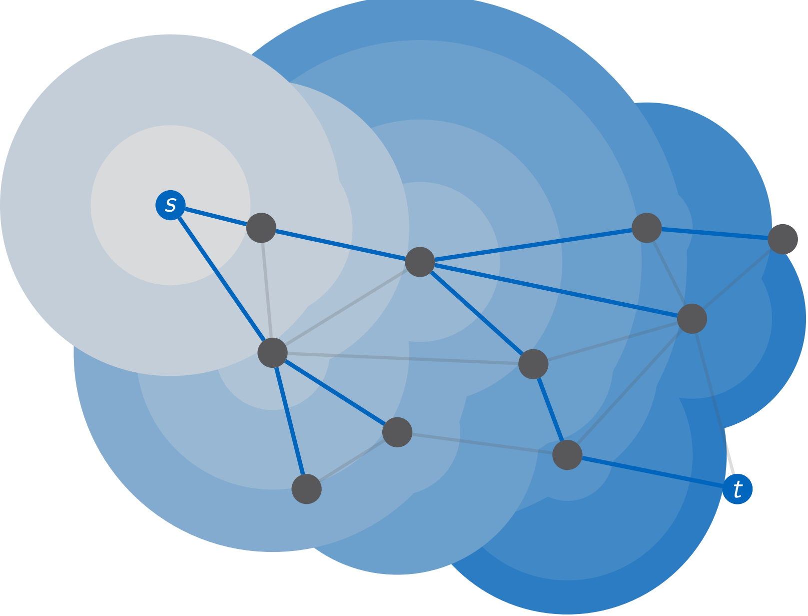
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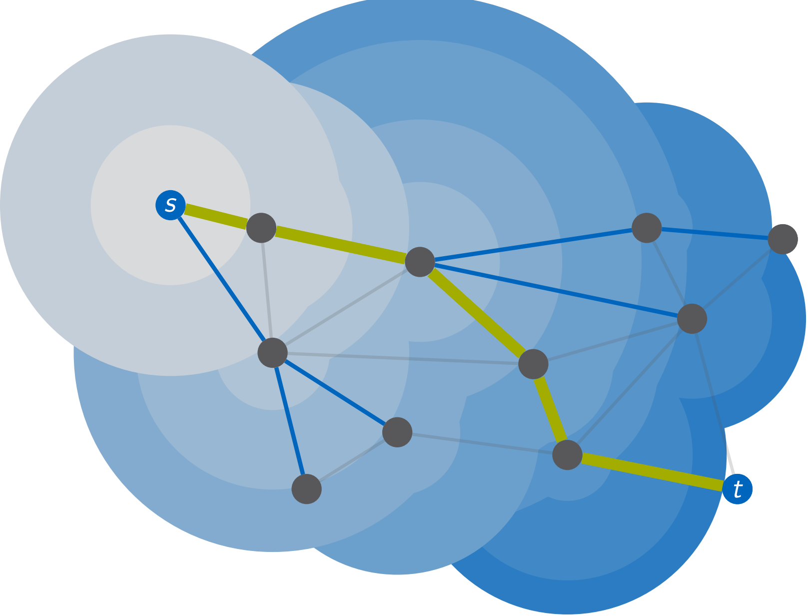
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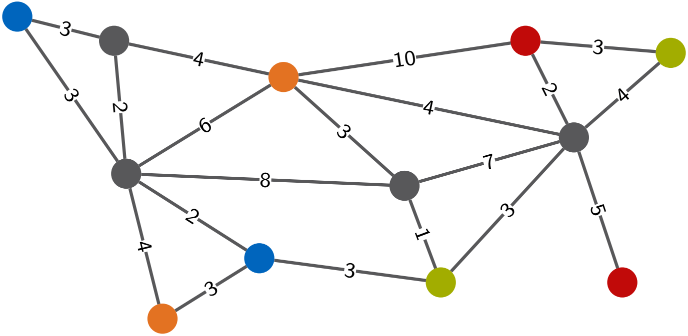
The generalized Steiner tree problem
(aka Steiner Forest)

Steiner Forest

Input: graph $G = (V, E)$, weights $w : E \to \mathbb{R}_+$,
terminal pairs $\{s_i, t_i\}$ for $i \in [n]$

Task: find $F \subseteq E$ containing an s_i-t_i-path for every $i \in [n]$
minimizing $\sum_{e \in F} w_e$



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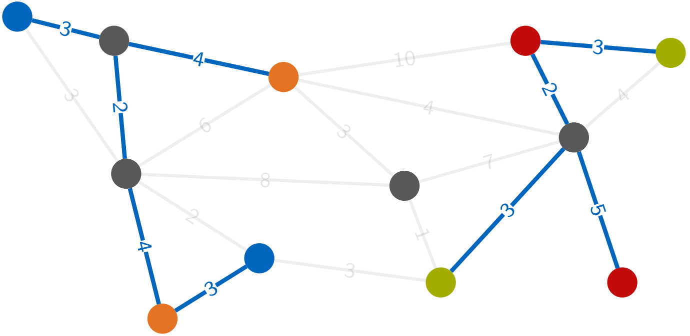
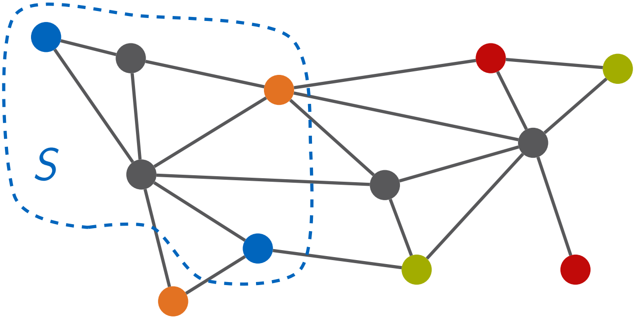
Task: find $F \subseteq E$ containing an s_i-t_i-path for every $i \in [n]$ minimizing $\sum_{e \in F} w_e$



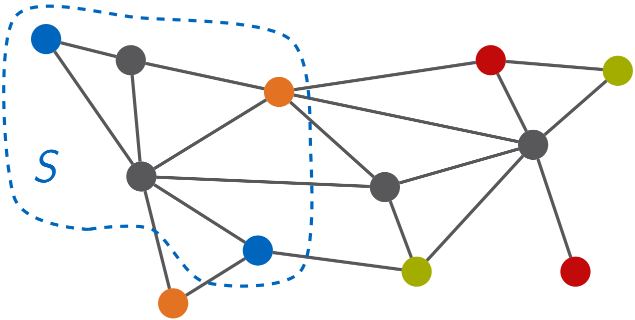
LP relaxation



$$\mathcal{S} := \{S \subseteq V \,:\, |S \cap \{s_i, t_i\}| = 1 \text{ for some } i \in [n]\}$$

$$\begin{aligned} \min \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq 1 \qquad \forall\, S \in \mathcal{S} \\ & x_e \in \{0,1\} \quad \forall\, e \in E \end{aligned}$$

LP relaxation



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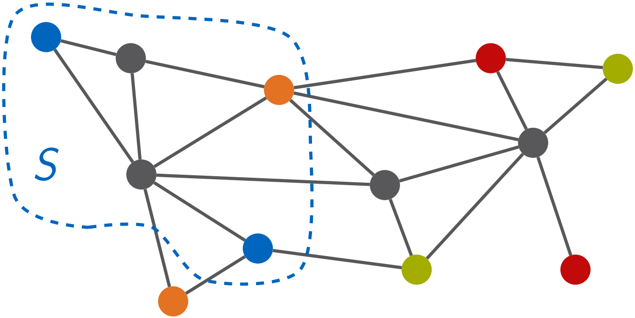

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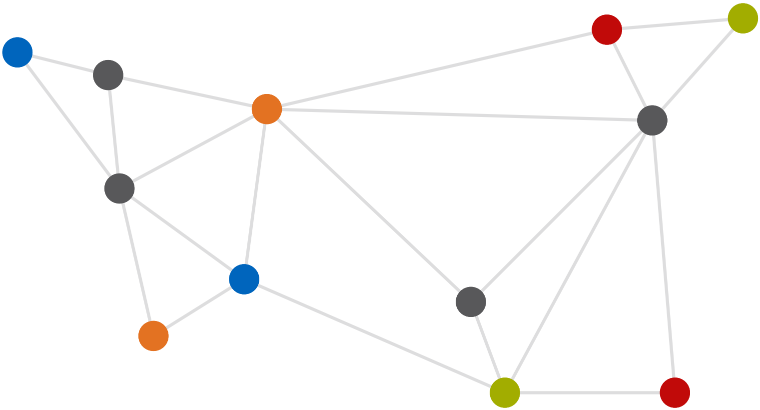
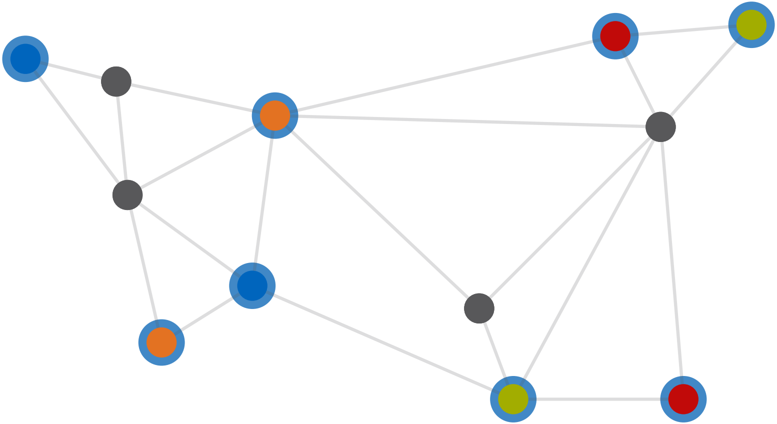
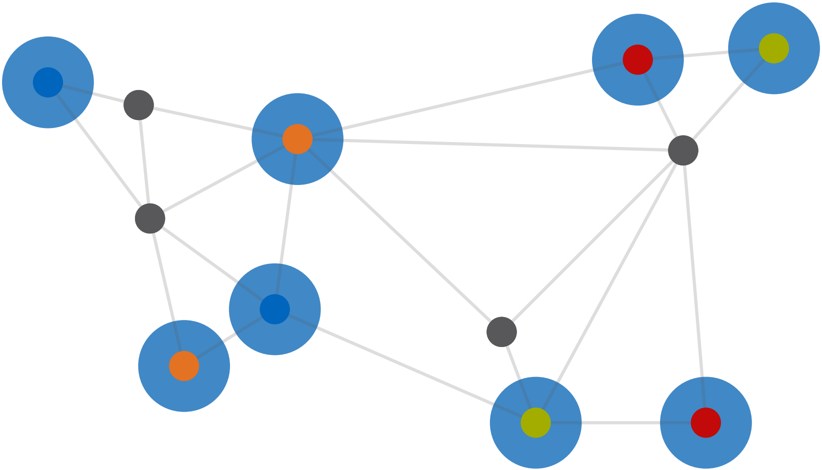
Primal-dual algorithm

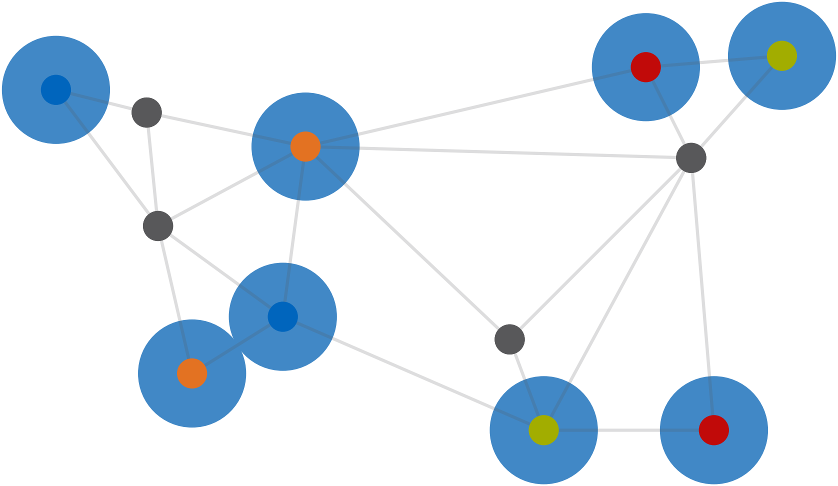
Algorithm

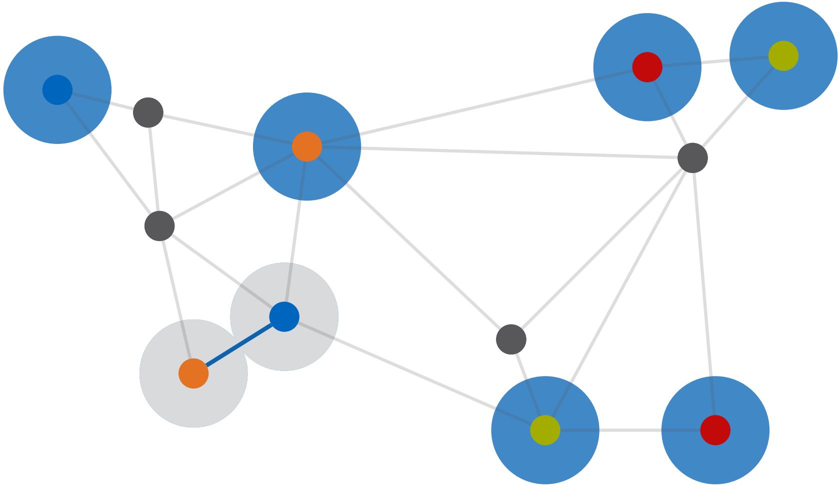
1. while $\exists\, i : F$ does not contain s_i-t_i-path
   - Let $\mathcal{C} := \{C \text{ conn. comp. of } F : |C \cap \{s_i, t_i\}| = 1 \text{ for some } i\}$.
   - Uniformly increase y_C for all $C \in \mathcal{C}$ until $\sum_{S \in \mathcal{S}_e} y_S = w_e$ for some $e \in \bigcup_{C \in \mathcal{C}} \delta(C)$. (increase can be 0)
   - $F := F \cup \{e\}$
2. $F := F \setminus \{e \in F : e \text{ is on no } s_i\text{-}t_i\text{-path in } F\}$
3. return F

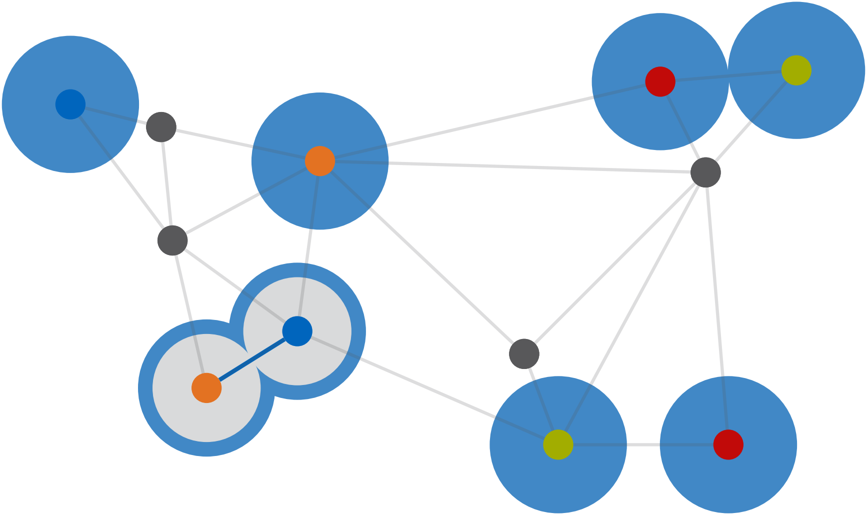


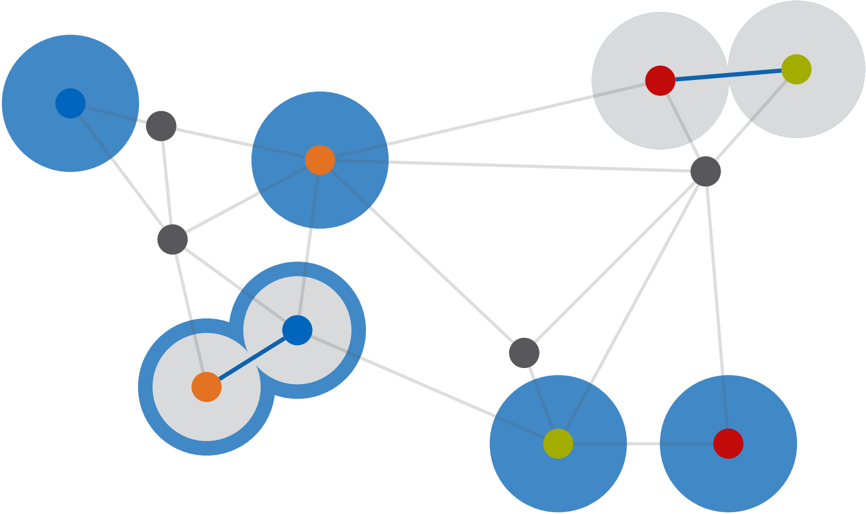


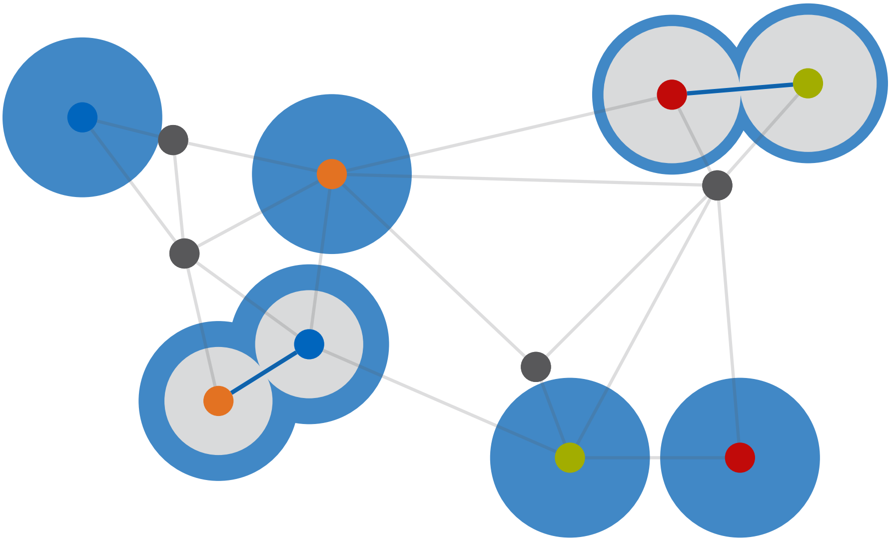


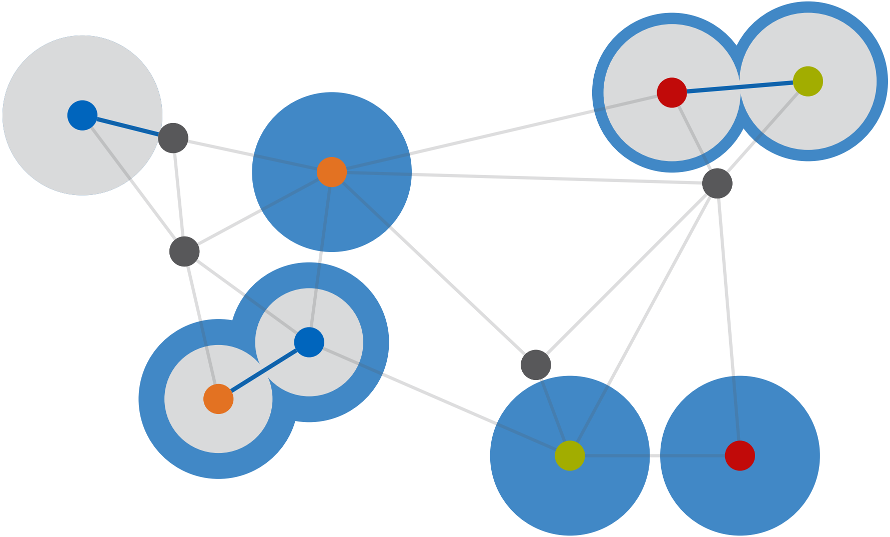


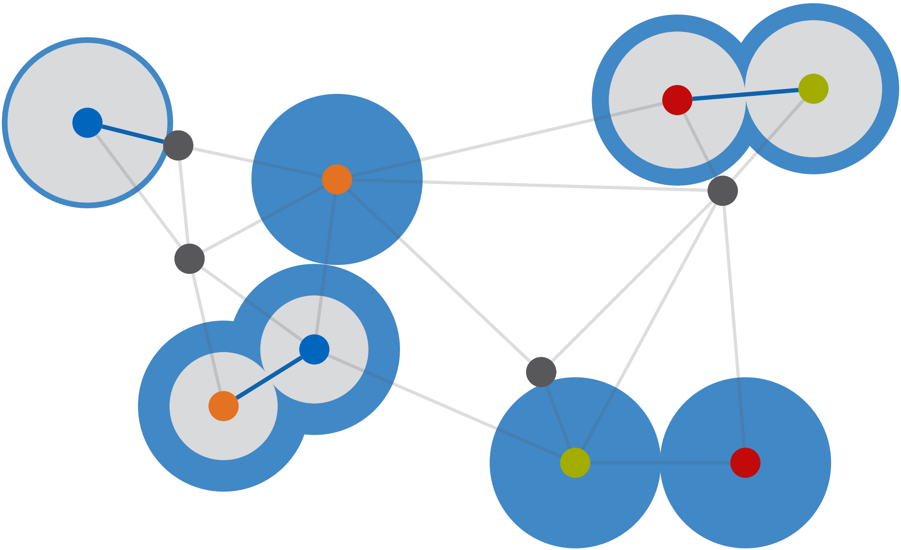


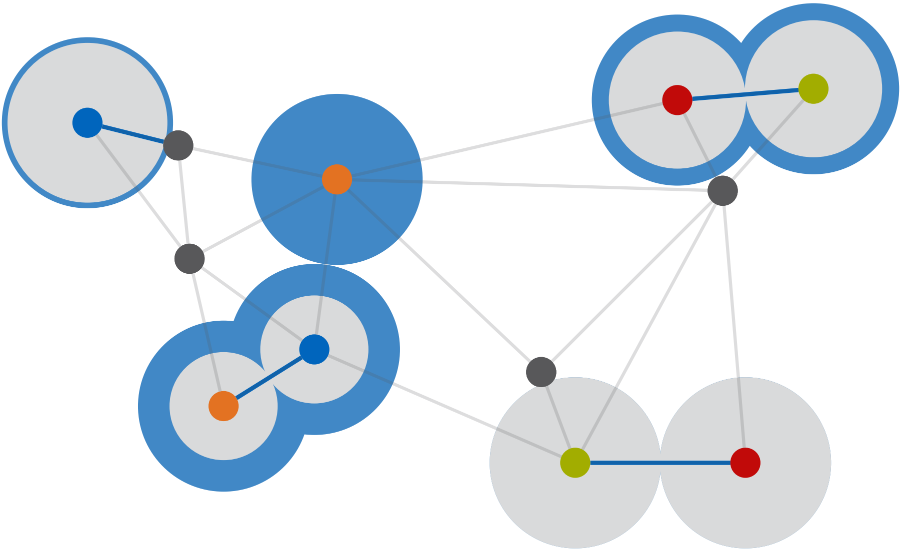


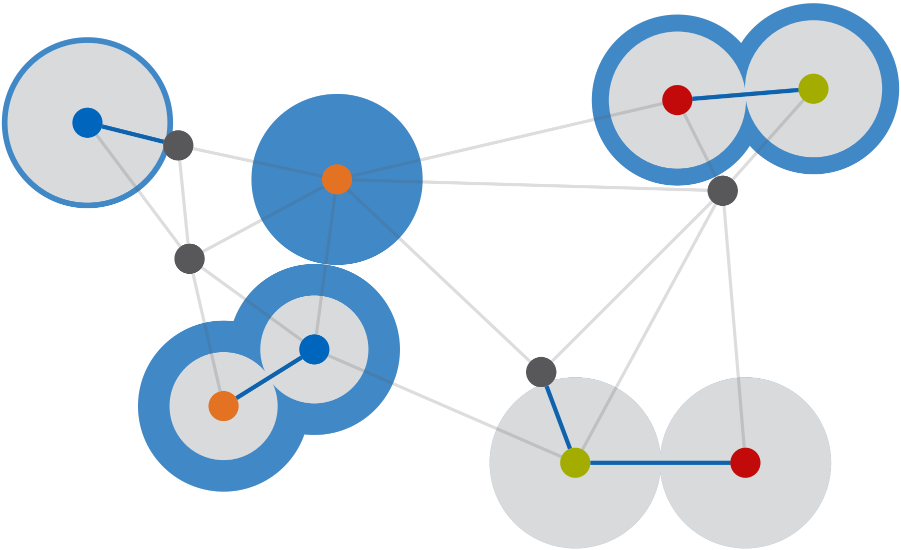


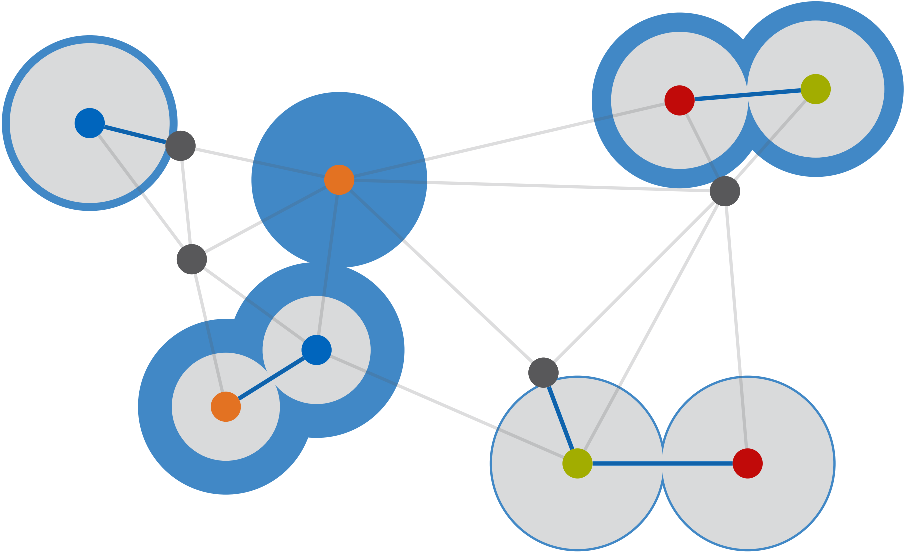


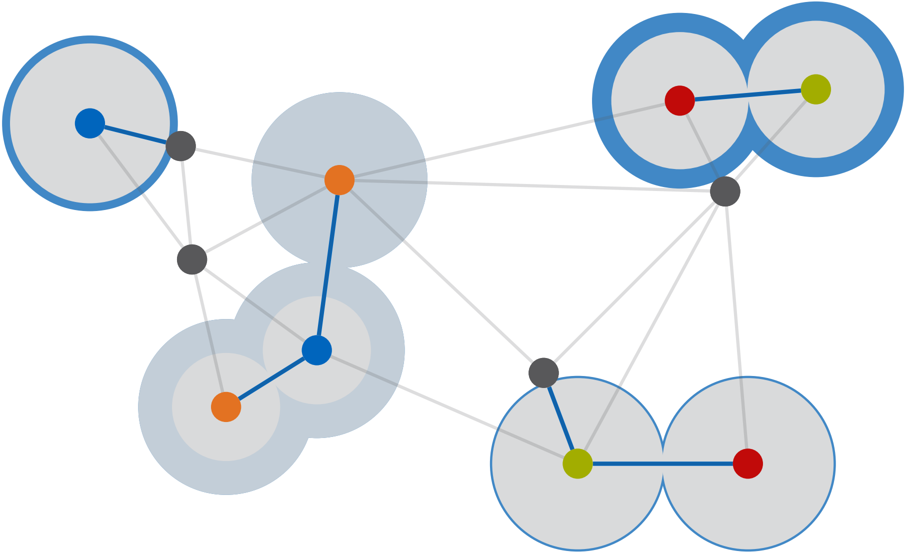


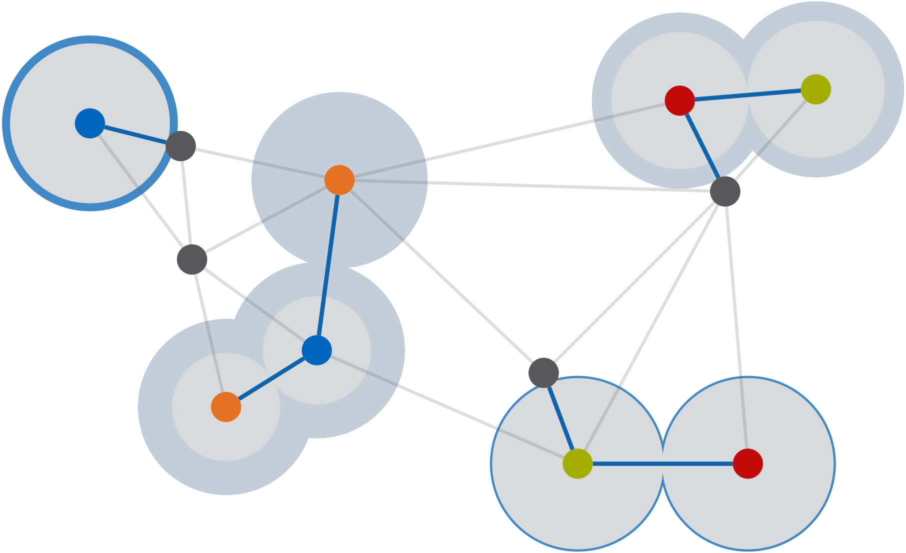


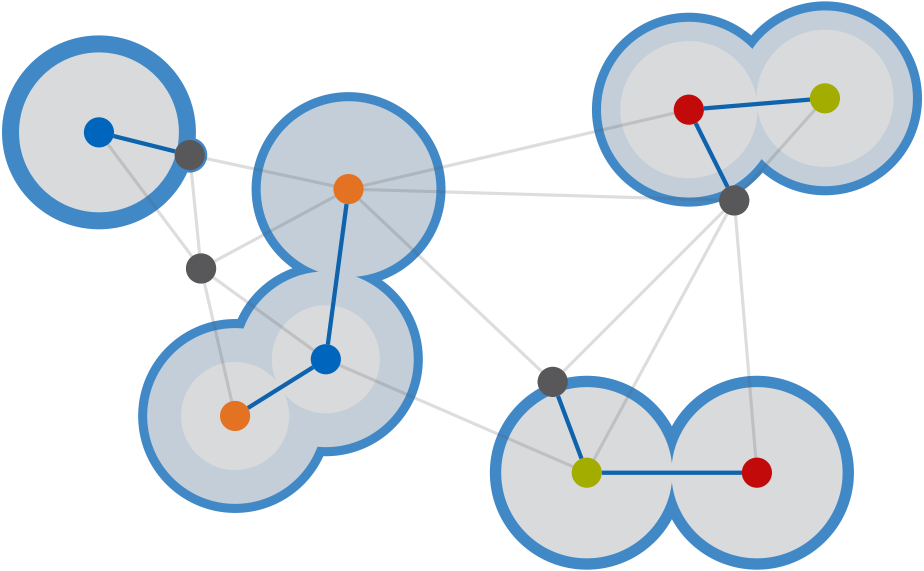


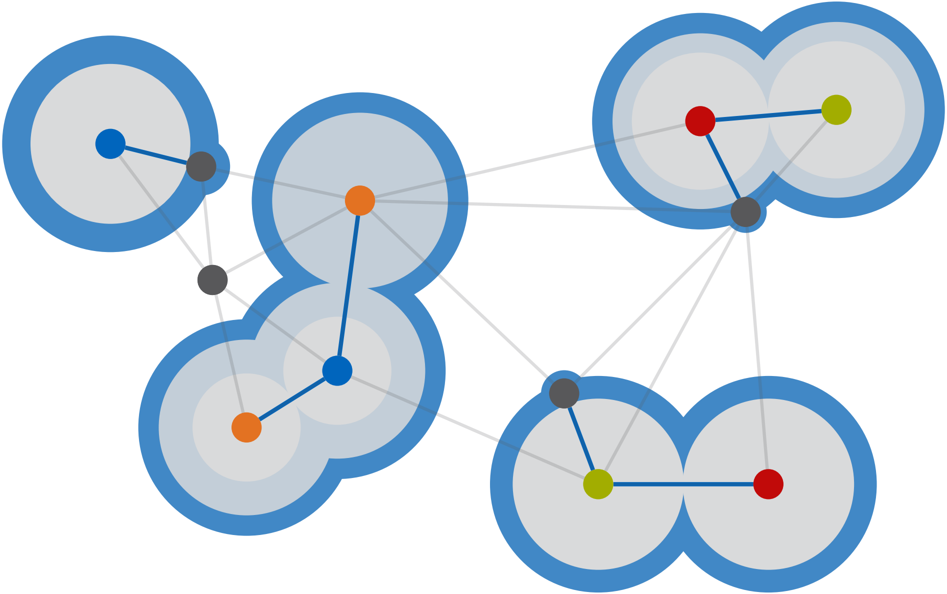


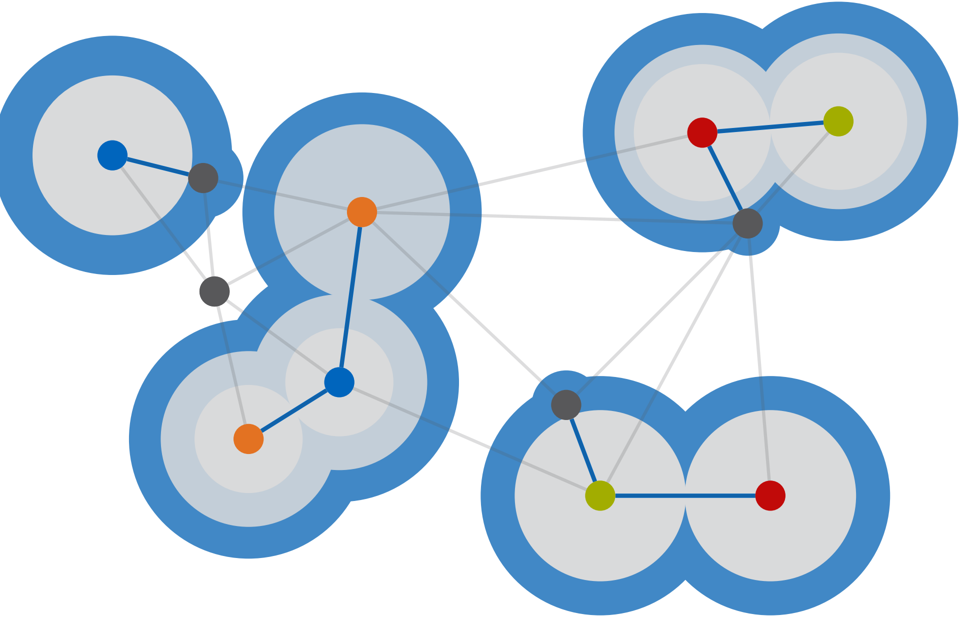


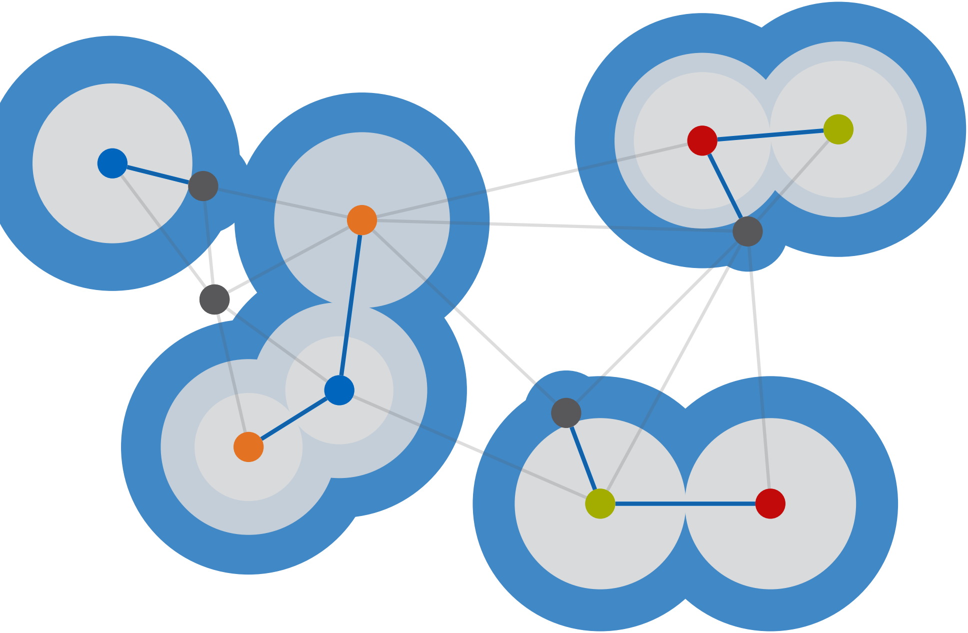


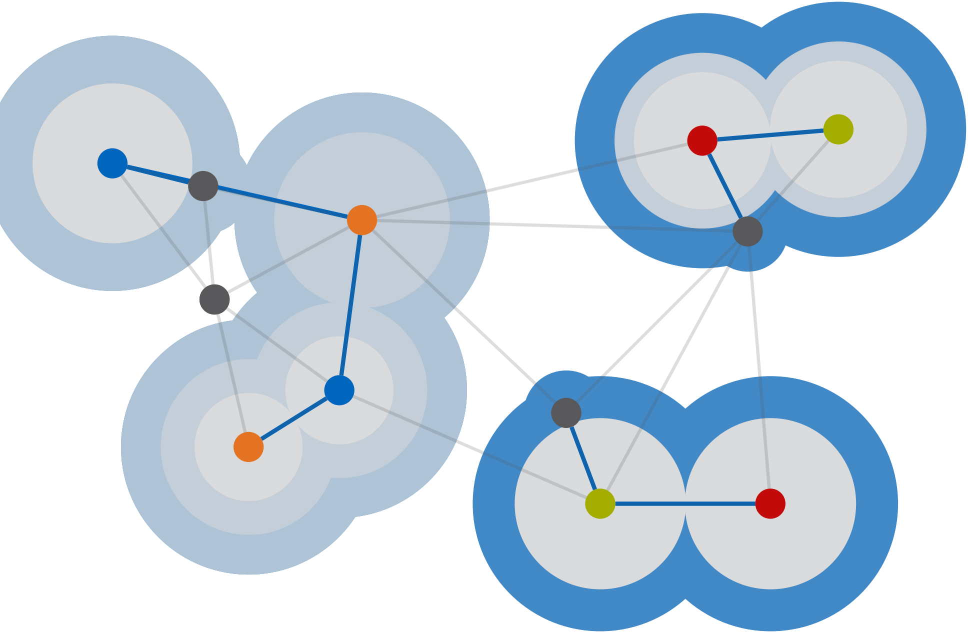


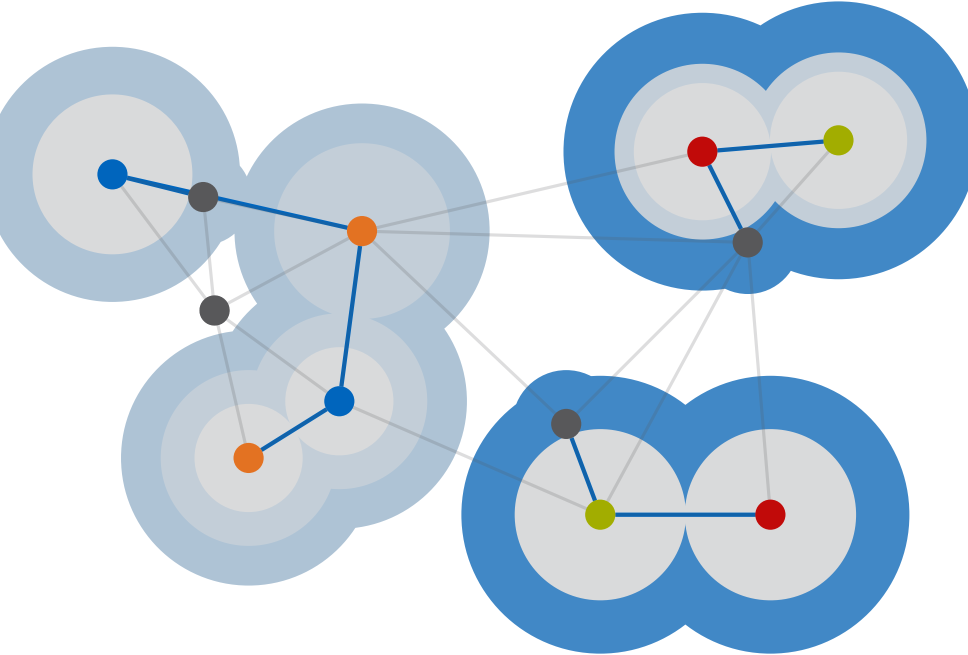


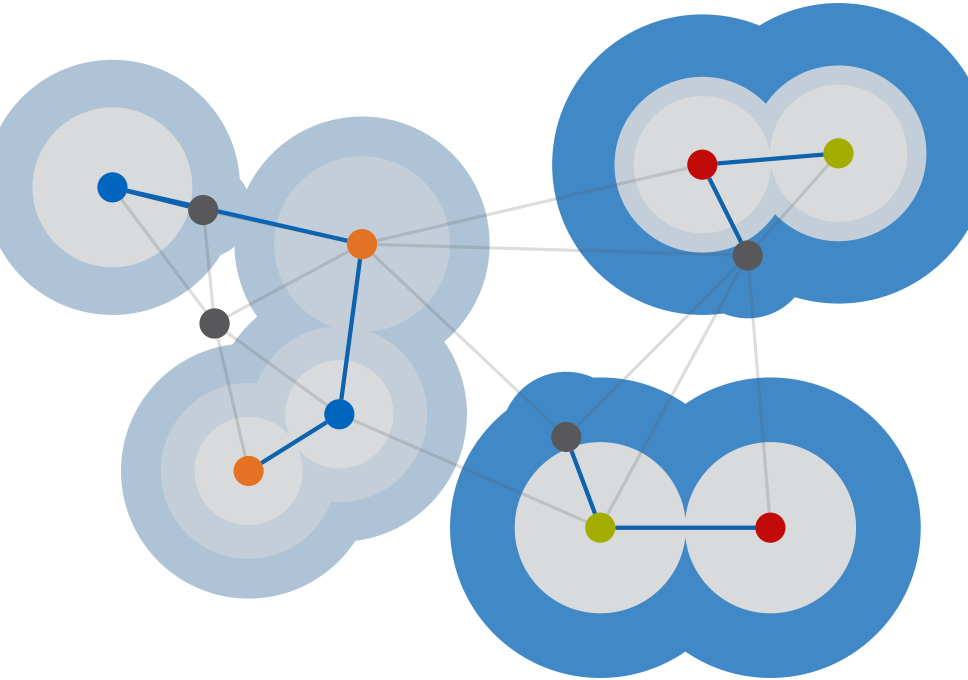


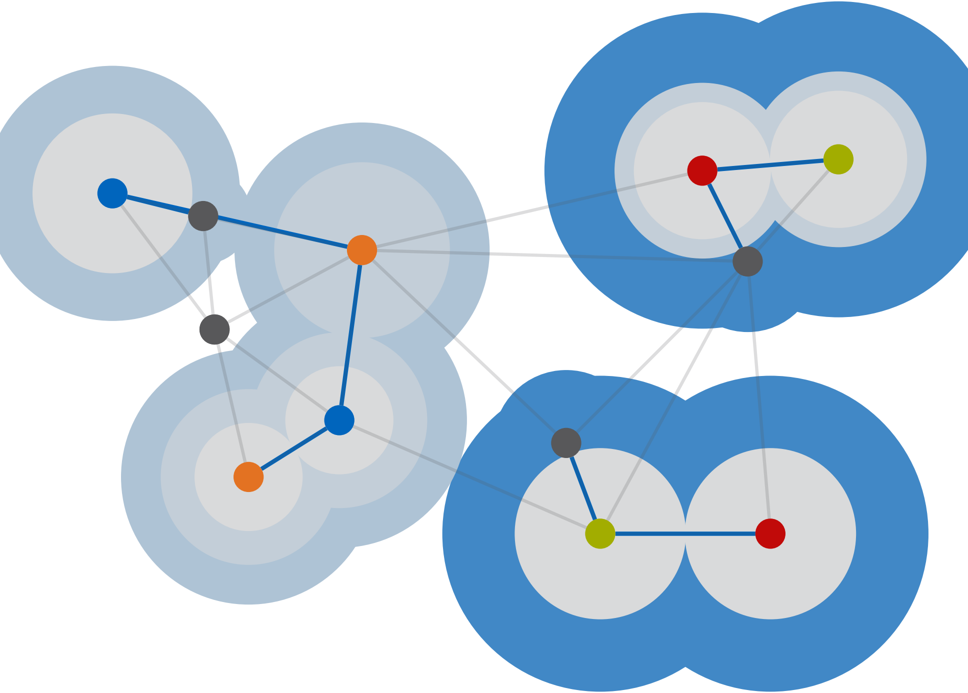


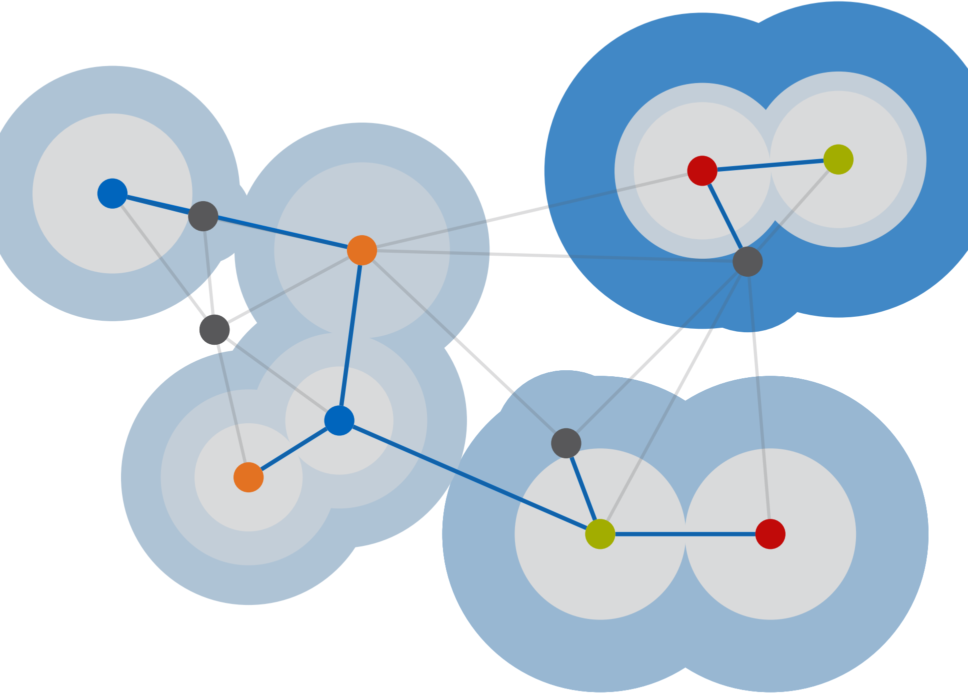


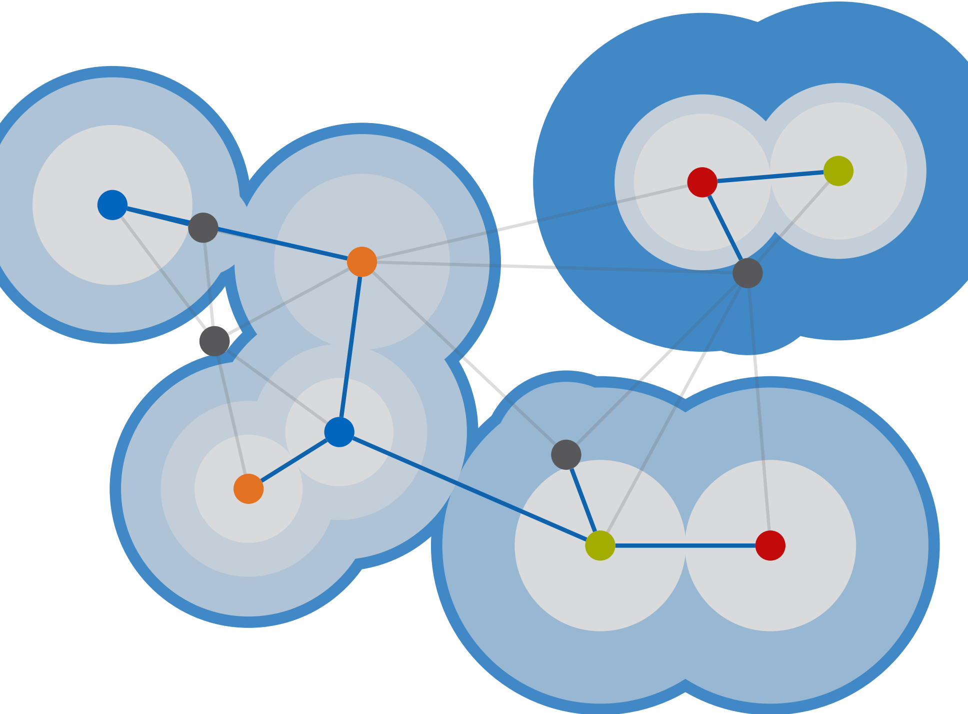


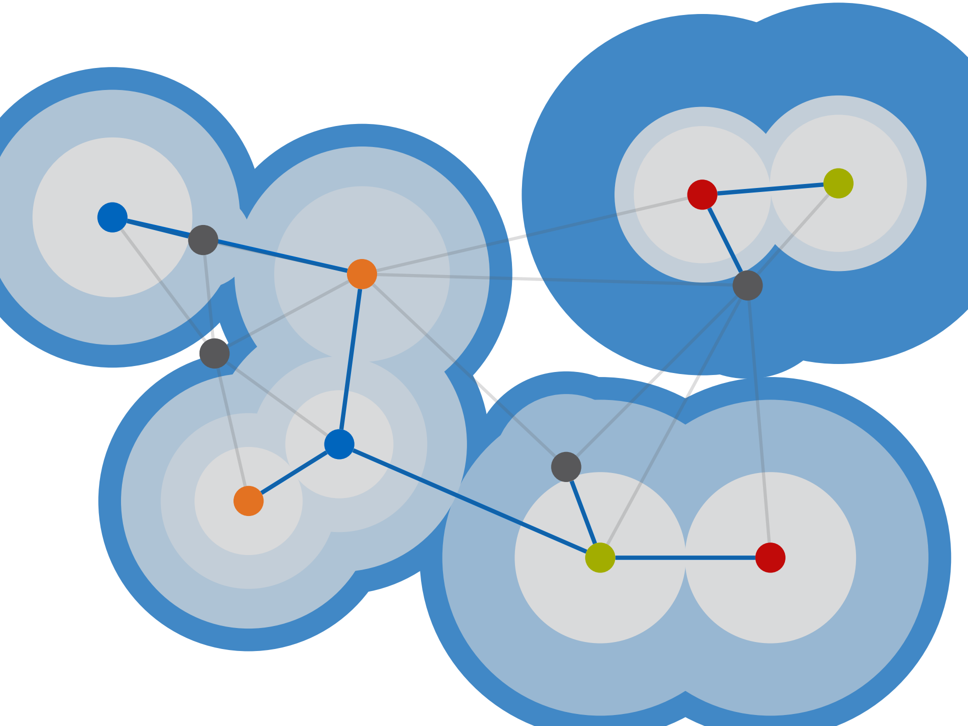


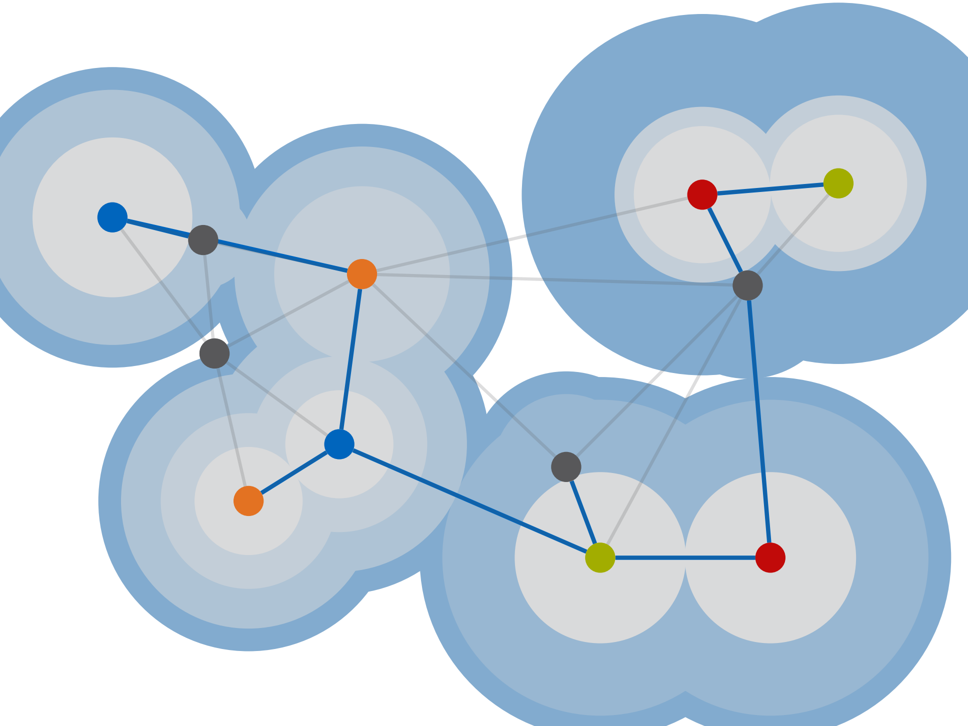


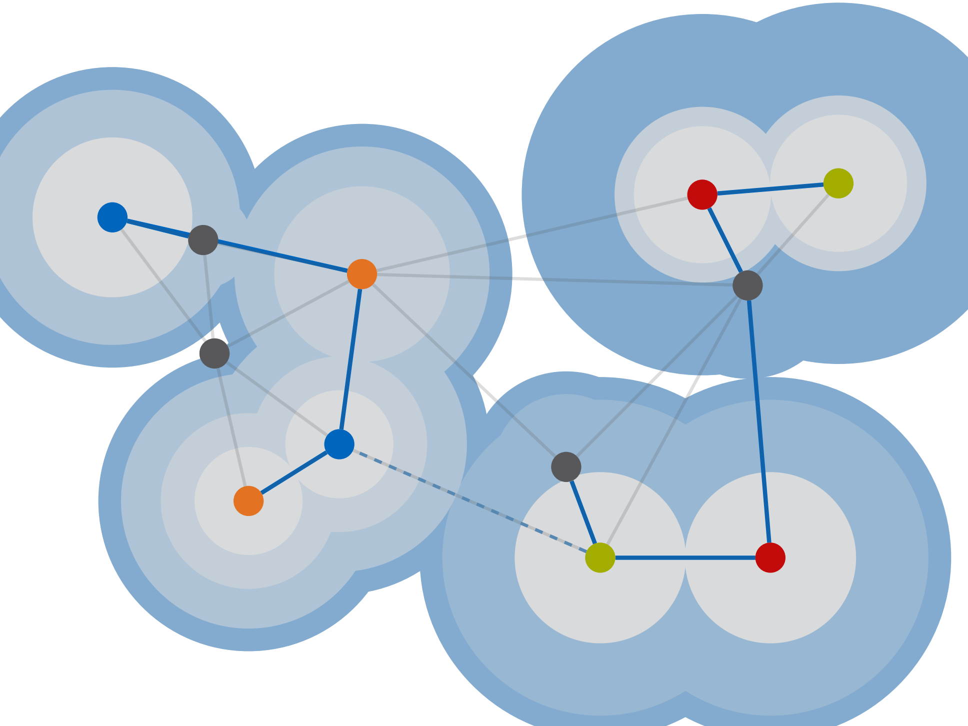


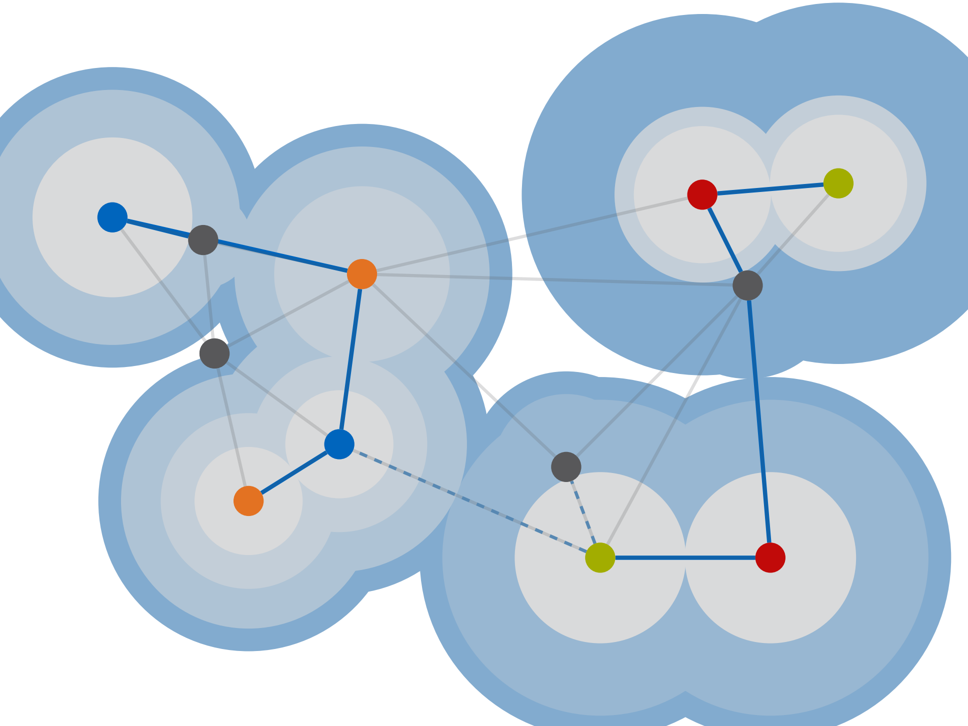












Primal-dual algorithm

Algorithm

1. while $\exists\, i : F$ does not contain s_i-t_i-path
   - Let $\mathcal{C} := \{C \text{ conn. comp. of } F : |C \cap \{s_i, t_i\}| = 1 \text{ for some } i\}$.
   - Uniformly increase y_C for all $C \in \mathcal{C}$ until $\sum_{S \in \mathcal{S}_e} y_S = w_e$ for some $e \in \bigcup_{C \in \mathcal{C}} \delta(C)$. (increase can be 0)
   - $F := F \cup \{e\}$
2. $F := F \setminus \{e \in F : e \text{ is on no } s_i\text{-}t_i\text{-path in } F\}$
3. return F

Theorem

Primal-dual is a 2-approximation algorithm for Steiner Forest.

Lemma

Let $\bar{F}$ be the tree returned by the alrogithm. At the beginning of every iteration: $\sum_{C \in \mathcal{C}} |\bar{F} \cap \delta(C)| \leq 2|\mathcal{C}|$

Happy holidays!