Technische Universität München, Zentrum Mathematik
Lehrstuhl für Operations Research

Advanced Graph Algorithms (MA5227)

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Problem Set 3

Homework Exercise 3.1 (Dealing with Multigraphs)
A *loop* is an arc with $\text{head}(a) = \text{tail}(a)$. Two arcs a, a' are *parallel* if $\text{head}(a) = \text{head}(a')$ and $\text{tail}(a) = \text{tail}(a')$. A digraph is *simple* if it does not contain any loops and pairs of parallel arcs.

a) Show that if $D = (V, A)$ is a simple digraph such that there is a node $s \in V$ such that every $v \in V$ is reachable from s, then $|V| - 1 \leq |A| \leq |V| \cdot (|V| - 1)$.

b) You bought a software from a shady algorithms dealer, who claimed that it solves the maximum flow problem very quickly. At home, you discover that, while the software is indeed very fast and computes a maximum flow, it only accepts as input instances of the maximum flow problem for which the underlying digraph is simple. What can you do to solve instances where this is not the case without increasing the size of the network?

c) You discover that the software also can solve the minimum cost flow problem, again with the limitation that it only works on simple graphs. Can you apply the same trick as in b)? Can you do something else maybe by slightly increasing the size of the network?

Homework Exercise 3.2 (Connecting Maximum Flows & b-Flows)
Consider the following problem: Given a network $D = (V, A)$ with capacities u and node balances b, either find a b-flow in D or correctly determine that no such flow exists. Show that this problem can be reduced to the maximum flow problem.
Hint: For your proof you may use the construction from the lecture.

Homework Exercise 3.3 (Augmenting b-Flows)
Let $D = (V, A)$ be a digraph, let $u : A \to \mathbb{R}_+$, let $c : A \to \mathbb{R}$ and let $b, b' \in \mathbb{R}^V$. Let x be a b-flow in D for capacities u and let x' be a b'-flow in D_x (the residual network of x) for capacities u_x. Extend x' by defining $x'(a) := 0$ for all $a \in \overleftrightarrow{A} \setminus D_x$. For all $a \in A$, define $x''(a) := x(a) + x'(a) - x'(\overleftarrow{a})$.

a) Show that x'' is a $(b + b')$-flow in D for capacities u.

b) Argue that a) implies the following two results from the lecture:

   i) Augmenting an s-t-flow x along an x-augmenting path P by $\gamma := \min_{a \in P} u_x(a)$ yields an s-t-flow of value $\text{val}(x) + \gamma$.

   ii) Augmenting a b-flow x along an x-augmenting cycle C by $\gamma := \min_{a \in C} u_x(a)$ yields a b-flow of cost $\text{cost}(x) + \gamma \sum_{a \in C} c(a)$.

Please turn over.

Tutorial Exercise 3.4 (Towards Finding the Minimum Mean Weight Cycle)
In order for the minimum mean cost cycle canceling algorithm to work, we need to be able to find such a cycle efficiently. The following insight turns out to be very helpful for this.

Let $D = (V, A)$ be a digraph with $n := |V|$ and $w : A \to \mathbb{R}$ and assume there is a node $s \in V$ such that each node $v \in V$ is reachable from s. For $v \in V$ and $k \in \mathbb{N}$ define

$$F_k(v) := \min \left\{ \sum_{i=1}^{k} w(a_i) : (a_1, \ldots, a_k) \text{ is an } s\text{-}v\text{-walk in } D \right\}$$

as the minimum weight of an s-v-walk of length exactly k in D.

Let $\mu(D, w) := \min \left\{ \frac{\sum_{a \in C} w(a)}{|C|} : C \text{ is a cycle in } D \right\}$ be the minimum mean weight cycle in D.

a) Show that if $\mu(D, w) = 0$, then

$$\min_{v \in V} \max_{0 \le k \le n-1} \frac{F_n(v) - F_k(v)}{n - k} = 0.$$

b) Use a) to show that

$$\mu(D, w) = \min_{v \in V} \max_{0 \le k \le n-1} \frac{F_n(v) - F_k(v)}{n - k}$$

is always true.

Remark: An s-v-walk in D is a sequence of arcs $(a_1, \ldots, a_k)$ such that $a_i \in A$ for all $i \in \{1, \ldots, k\}$, $\text{tail}(a_1) = s$, $\text{head}(a_k) = v$ and $\text{head}(a_i) = \text{tail}(a_{i+1})$ for $i \in \{1, \ldots, k-1\}$. Note that this definition allows the walk to use arcs multiple times. Furthermore, we introduce the convention $\min \emptyset := \infty$.

This problem set will be discussed in the tutorials on December 4th & 5th, 2017.