Proof of Lemma 12.1: Let M' be a perfect matching. Then:

$$\sum_{e \in M'} w(e) \geq \sum_{e \in M'} \sum_{A \in \mathcal{A} : e \in \delta(A)} z(A) \geq \sum_{A \in \mathcal{A}} z(A) = \sum_{e \in M} \sum_{A \in \mathcal{A} : e \in \delta(A)} z(A) = \sum_{e \in M} w(e) \qquad \square$$

$|\delta(A) \cap M| \geq 1 \;\forall A \in \mathcal{A}$
$|\delta(v) \cap M| = 1 \;\forall v \in V$

$z(A) > 0 \Rightarrow |\delta(A) \cap M| = 1$

$e \in M \Rightarrow e$ is z-tight

Invariants of the Weighted Matching Algorithm

(i) M is a matching and z is a feasible solution to [D]

(ii) $e \in M \Rightarrow e$ is z-tight (also true for $e \in E(F)$)

(iii) $z(A) > 0 \Rightarrow A$ is a blossom of M (in particular, $|\delta(A) \cap M| \leq 1$)

(iv) $z(A), z(B) > 0 \Rightarrow A \cap B = \emptyset$ or $A \subseteq B$ or $B \subseteq A$
("support of z is laminar")

Consequences:

(i) - (iii) $\Rightarrow$ M is min weight perfect matching at the end of the algorithm

(iv) $\Rightarrow$ Case 3 of DUAL can only happen $O(|V|)$ times between two AUGMENT steps.

- Case 3 only affects odd vertices, so must have been shrunk before last AUGMENT.
- Laminar family on V can only consist of $O(|V|)$ sets.

Details: Korte-Vygen Sec. 11.2-11.3

Theorem 12.3 The Gale-Shapley Algorithm returns a stable matching in time $O(|A||B|)$.

Proof: Note that throughout the algorithm

- $\mu(a)$ never decreases w.r.t. $\prec_a$ for any $a \in A$,
- $\mu(b)$ never increases w.r.t. $\prec_b$ for any $b \in B$.

In every iteration of the while-loop, $\mu(a)$ increases for one $a \in A$. Therefore algorithm terminates after at most $|A| \cdot |B|$ iterations.

Next: Show that $M := \{\{a, \mu(a)\} : a \in A, \mu(a) \neq \emptyset\}$ is a stable matching.

At termination: $\mu(a) \neq \emptyset$ implies $\mu(\mu(a)) = a$ (by while condition)

$\Rightarrow M$ is a matching.

Furthermore, $\mu(b) \neq \emptyset$ implies $\mu(\mu(b)) = b$ (show by induction on the algorithm).

Assume there is a blocking edge $\{a, b\} \in E \setminus M$.

Then (i) $\mu(a) \succ_a b$ and (ii) $\mu(b) \succ_b a$.

By (i) there was an iteration of the while loop when $\mu(a)$ was set to b. After the end of that iteration, $\mu(b) \preceq_b a$.

Because $\mu(b)$ only decreases, this contradicts (ii). $\square$